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Solution of the Angles-Only Satellite Tracking Problem

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INTRODUCTION

The determination of the orbit of a satellite vehicle or ballistic missile is obviously of great practical importance. In the usual case, the azimuth, elevation, range, and range-rate of the target are measured from an Earth-fixed radar site. When range and range-rate are not available, the determination of the orbit from the azimuth(s), elevation(s), and the time(s) of the observation is referred to as the "angles-only problem."

A good summary of this problem is treated in reference 1. In that source, it is pointed out that the problem was solved by Gauss, Laplace, and Escobal. These solutions seem to be far better known among astronomers than among aerospace engineers.*

In the present report, another solution is derived which is particularly well suited to numerical calculations. It is shown that all of the orbital elements (semi-major axis (a), eccentricity (e), argument of perigee ($\tilde{\omega}$), inclination (I), and longitude of the ascending node (Ω)) can be derived from five measurements of the azimuth (and the times of observation), or five measurements of the elevation (and the times of observation). Once these elements have been isolated, the epoch can be derived from a given time measurement.

An advantage of the present formulation is that it can be readily extended to the case of an observer aboard a second satellite rather than an Earth-fixed observer. Since only line-of-sight angles would be required to determine the orbit of a second satellite, the technique could be used to eliminate range and range-rate hardware that is presently required for rendezvous. This option will be treated in a future paper.

THE DRIVER PROGRAM

In the case of an actual radar installation, the azimuth and elevation angles, as well as the time of the observations, would be recorded once a target is acquired. Since this report deals with theoretical sightings rather than actual observations, it is necessary to build a driver program to simulate what a real radar station would observe. To that end, a vehicle having arbitrary orbital elements is assumed and the details of the sightings from a general configuration are derived. From a general configuration, specific orbital elements are used for simulation. Once the elevation, azimuth, and time are calculated, the orbital parameters are hidden from the isolation scheme. The isolation then has to rederive a complete specification of the orbit from the table of elevation angle versus time (or the azimuth versus time). Since a ballistic missile that impacts the Earth is obviously in a "terminated" orbit, the same theory applies in either case, i.e., one can track an incoming missile using the present theory. For the ballistic missile case, drag would be a significant factor, however. Drag is not included in the present report.

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* The author wishes to thank Dr. John Hanson, EL-58, MSFC, for pointing out this literature on this subject.

Let $(\hat{\imath}, \hat{\jmath}, \hat{k})$ describe an Earth-centered inertial coordinate system and locate a tracking station by the polar coordinates (R_o, ϕ_o, θ_o) (fig. 1). We explicitly recognize the Earth's rotation by setting $\phi_o = \phi_o(t)$. At the tracking station, a spherical coordinate system is constructed via the unit vectors $(\hat{u}_r, \hat{u}_\theta, \hat{u}_\phi)$. Note that this coordinate system is modified from the usual convention in that the $\hat{u}_\theta$ points due south along the meridian rather than the conventional practice of orienting it to the north. The target is located, relative to the tracking station by a distance, ρ, and two angles, the azimuth α, and the elevation angle, β.

The unit vector transformation is:

$$\hat{u}_r = \cos\theta_o \cdot \cos\phi_o \cdot \hat{\imath} + \cos\theta_o \cdot \sin\phi_o \cdot \hat{\jmath} + \sin\theta_o \cdot \hat{k},$$

$$\hat{u}_\phi = \sin\phi_o \cdot \hat{\imath} + \cos\phi_o \cdot \hat{\jmath},$$

$$\hat{u}_\theta = \sin\theta_o \cdot \cos\phi_o \cdot \hat{\imath} + \sin\theta_o \cdot \sin\phi_o \cdot \hat{\jmath} - \cos\theta_o \cdot \hat{k},$$

along with the vector from the center of the Earth to the radar station, $\vec{R}_o$, as

$$\vec{R}_o = R_o \cdot \cos\theta_o \cdot \cos\phi_o \cdot \hat{\imath} + R_o \cdot \cos\theta_o \cdot \sin\phi_o \cdot \hat{\jmath} + R_o \cdot \sin\theta_o \cdot \hat{k} \ , \tag{1}$$

where $R_o = |\vec{R}_o|$.

In the local coordinates of the radar station, the ρ vector is given by

$$\vec{\rho} = \rho \cdot (\sin\beta \cdot \hat{u}_r + \cos\beta \cdot \sin\alpha \cdot \hat{u}_\phi + \cos\beta \cdot \cos\alpha \cdot \hat{u}_\theta), \tag{2}$$

where $\rho = |\vec{\rho}\,|$.

Inserting the expressions for $\hat{u}_r$, $\hat{u}_\phi$, and $\hat{u}_\theta$ into the expressions for $\vec{\rho}$, one finds

$$\begin{aligned}\vec{\rho} = \rho \cdot [&(\sin\beta \cdot \cos\theta_o \cdot \cos\phi_o - \cos\beta \cdot \sin\alpha \cdot \sin\phi_o + \cos\beta \cdot \cos\alpha \cdot \sin\theta_o \cdot \cos\phi_o) \cdot \hat{\imath} \\ &+ (\sin\beta \cdot \cos\theta_o \cdot \sin\phi_o + \cos\beta \cdot \sin\alpha \cdot \cos\phi_o + \cos\beta \cdot \cos\alpha \cdot \sin\theta_o \cdot \sin\phi_o) \cdot \hat{\jmath} \\ &+ (\sin\beta \cdot \sin\theta_o - \cos\beta \cdot \cos\alpha \cdot \cos\theta_o) \cdot \hat{k}] \ .\end{aligned} \tag{3}$$

Define a vector $\vec{r}$ from the center of the Earth to the satellite. For reference 2, one can write $\vec{r}$ in terms of the inclination, I, the right ascension of the ascending node, Ω, the argument of perigee, ϖ, and the true anomaly, f, as

$$\begin{aligned}\vec{r} = x \cdot \hat{\imath} + y \cdot \hat{\jmath} + z \cdot \hat{k} = &\, r \cdot [\cos(f+\tilde{\omega}) \cdot \cos\Omega - \sin(f+\tilde{\omega}) \cdot \sin\Omega \cdot \cos I] \cdot \hat{\imath} \\ &+ r \cdot [\cos(f+\tilde{\omega}) \cdot \sin\Omega + \sin(f+\tilde{\omega}) \cdot \cos\Omega \cdot \cos I] \cdot \hat{\jmath} \\ &+ r \cdot \sin(f+\tilde{\omega}) \cdot \sin I \cdot \hat{k} \ ,\end{aligned} \tag{4}$$

where $r = |\vec{r}\,|$ (see fig. 2). Also, one can write

$$r = a \cdot (1-e^2)/(1+e \cdot \cos f) = p/(1+e \cdot \cos f) \ , \tag{5}$$

where a is the semi-major axis of the orbit and e as the eccentricity and $p = a\cdot(1-e^2)$.

Vectorially,

$$\vec{r} = \vec{R}_o + \vec{\rho} \ . \tag{6}$$

Equating the components of equation (6) from equations (5), (3), and (1),

$$\rho\cdot(\sin\beta\cdot\cos\theta_o\cdot\cos\phi_o - \cos\beta\cdot\sin\alpha\cdot\sin\phi_o + \cos\beta\cdot\cos\alpha\cdot\sin\theta_o\cdot\cos\phi_o)$$

$$= r\cdot[\cos(f+\tilde{\omega})\cdot\cos\Omega - \sin(f+\tilde{\omega})\cdot\sin\Omega\cdot\cos I] - R_o\cdot\cos\theta_o\cdot\cos\phi_o \ , \tag{7}$$

$$\rho\cdot(\sin\beta\cdot\cos\theta_o\cdot\sin\phi_o + \cos\beta\cdot\sin\alpha\cdot\cos\phi_o + \cos\beta\cdot\cos\alpha\cdot\sin\theta_o\cdot\sin\phi_o)$$

$$= r\cdot[\cos(f+\tilde{\omega})\cdot\sin\Omega + \sin(f+\tilde{\omega})\cdot\cos\Omega\cdot\cos I] - R_o\cdot\cos\theta_o\cdot\sin\phi_o \ , \tag{8}$$

$$\rho\cdot(\sin\beta\cdot\sin\theta_o - \cos\beta\cdot\cos\alpha\cdot\cos\theta_o) = r\cdot\sin(f+\tilde{\omega})\cdot\sin I - R_o\cdot\sin\theta_o \ . \tag{9}$$

Taking the square root of the sum of the squares of equations (7), (8), and (9) gives

$$\rho = \langle r^2 + R_o^2 - 2\cdot r\cdot R_o\cdot\{[\cos(f+\tilde{\omega})\cdot\cos\theta_o\cdot\cos(\Omega-\phi_o) - \sin(f+\tilde{\omega})]\cdot[\cos\theta_o\cdot\cos I\cdot\sin(\Omega-\phi_o)$$

$$-\sin I\cdot\sin\theta_o]\}\rangle^{1/2}. \tag{10}$$

Although the preceding equations can be solved directly for α and β, a more subtle approach yields additional quadrant information that can be valuable during numerical calculations. To this end, one first calculates (8)$\cdot\cos\phi_o$–(7)$\cdot\sin\phi_o$ to get

$$\rho\cdot\cos\beta\cdot\sin\alpha = r\cdot[\cos(f+\tilde{\omega})\cdot\sin(\Omega-\phi_o) + \sin(f+\tilde{\omega})\cdot\cos I\cdot\cos(\Omega-\phi_o)] \ . \tag{11}$$

Next, calculate (7)$\cdot\cos\phi_o$+(8)$\cdot\sin\phi_o$ to yield

$$\rho\cdot(\sin\beta\cdot\cos\theta_o + \cos\beta\cdot\cos\alpha\cdot\sin\theta_o)$$

$$= r\cdot[\cos(f+\tilde{\omega})\cdot\cos(\Omega-\phi_o) - \sin(f+\tilde{\omega})\cdot\cos I\cdot\sin(\Omega-\phi_o) - R_o\cos\theta_o] . \tag{12}$$

From (12)$\cdot\sin\theta_o$–(9)$\cdot\cos\theta_o$ one has

$$\rho\cdot\cos\beta\cdot\cos\alpha = r\cdot\{\cos(f+\tilde{\omega})\cdot\cos(\Omega-\phi_o)\cdot\sin\theta_o$$

$$-\sin(f+\tilde{\omega})\cdot[\cos I\cdot\sin(\Omega-\phi_o)\cdot\sin\theta_o + \sin I\cdot\cos\theta_o]\} \ . \tag{13}$$

Now, divide equation (11) by equation (13) and obtain the formula for the tangent of α as

$$\tan\alpha = [\cos(f+\tilde{\omega})\cdot\sin(\Omega-\phi_o) + \sin(f+\tilde{\omega})\cdot\cos I\cdot\cos(\Omega-\phi_o)]/\{\cos(f+\tilde{\omega})\cdot\cos(\Omega-\phi_o)\cdot\sin\theta_o$$

$$-\sin(f+\tilde{\omega})\cdot[\cos I\cdot\sin(\Omega-\phi_o)\cdot\sin\theta_o + \sin I\cdot\cos\theta_o]\} \ . \tag{14}$$

Equation (14) may be rewritten in many forms, with one of the more useful being

$$\tan\alpha = [\sin(\delta\Omega-\phi_o)+\tan(f+\tilde{\omega})\cdot\cos I\cdot\cos(\delta\Omega-\phi_o)]/\langle\cos\theta_o\cdot\{\cos(\delta\Omega-\phi_o)\cdot\tan\theta_o$$

$$-\tan(f+\tilde{\omega})\cdot[\cos I\cdot\sin(\delta\Omega-\phi_o)\cdot\tan\theta_o+\sin I]\}\rangle\ . \tag{15}$$

In equation (14), one can separately interrogate the numerator and denominator (interpreted as $\sin\alpha$ and $\cos\alpha$) to obtain the needed quadrant information on α.†

The elevation, β, can be obtained in several ways. One such method would come from multiplying equation (12) by $\cos\theta_o$ and equation (9) by $\sin\theta_o$ and adding to obtain

$$\rho\cdot\sin\beta = r\cdot\{\cos(f+\tilde{\omega})\cdot\cos(\delta\Omega-\phi_o)\cdot\cos\theta_o-\sin(f+\tilde{\omega})\cdot[\cos I\cdot\sin(\delta\Omega-\phi_o)\cdot\cos\theta_o$$

$$-\sin I\cdot\sin\theta_o]\}-R_o. \tag{16}$$

Squaring and adding equations (11) and (13) yields a formula for $\cos\beta$, but this process destroys information. Dividing equation (15) by that equation for $\cos\beta$ will yield a tangent formula which involves r and R_o but not ρ.

Another approach to determine β is simply to utilize equation (10) to determine ρ and then use either equation (11) or equation (13) to obtain β as an arc cosine. Equation (11) would be greatly preferred since $\cos\alpha$ will usually vanish during a satellite pass.

The magnitude of the radius vector, r, which appears in many of the above equations involves the semi-major axis, a, and the eccentricity, e. If one has full knowledge of the orbital elements a, e, I, ϖ, and $\delta\Omega$, and if one also knows the epoch (or a related time point), one can now obtain the values of ρ, α, and β. The time reference serves to determine the true anomaly, f, and the position of the observation site, since $\phi_o = \phi_o(t)$.

For given values of the orbital elements and epoch (needed for r and ϕ_o), the path of computation is now clear. Equation (10) yields the values of ρ, and equation (15) can be solved for α. Once ρ and α are known, equation (11) provides the value of β.

It is apparent that not all values of the epoch will yield values of (ρ, α, β) which correspond to physical data that could be recorded at a given observation site. Subsidiary conditions must be imposed on the epoch (as well as on θ_o) in order to guarantee that the satellite is above the horizon of a ground plane which is tangent to the site. ("Over the horizon radar" is not considered here, but such a device could easily be incorporated into these equations.) Physically, apparition of the target is expected when $\beta = 0$ and occultation when $\beta = \pi$. These conditions will be fulfilled if

$$\vec{\rho}\bullet\vec{R}_o = 0\ .$$

Dotting equation (1) with equation (3) gives

$$\cos(f+\tilde{\omega})\cdot\cos(\delta\Omega-\phi_o)\cdot\cos\theta_o-\sin(f+\tilde{\omega})\cdot[\cos I\cdot\sin(\delta\Omega-\phi_o)\cdot\cos\theta_o-\sin I\cdot\sin\theta_o] = R_o/r\,. \tag{17}$$

This equation is time dependent through the variables f, r, and ϕ_o. In order to recognize the explicit time dependence of the longitude of the observation site, write

$$\phi_o \rightarrow \phi_o+\dot{\phi}\cdot t. \tag{18}$$

† The author wishes to thank Dr. Larry Mullins, MSFC, NASA, for pointing out the existence of this formula.

Since the true anomaly, f, is time dependent, r must be expanded by equation (5). In terms of the eccentric anomaly, $\mathcal{E}$,

$$r = a \cdot [1 - e \cdot \cos(\mathcal{E})]. \tag{19}$$

The eccentric anomaly is related to the time by Gauss' equation (zero referenced to the time of perigee passage) as

$$t = \sqrt{(a^3/\mu)} \cdot (\mathcal{E} - e \cdot \sin \mathcal{E}), \tag{20}$$

and to the true anomaly, f, via

$$f = 2 \cdot \tan^{-1}\{\sqrt{[(1+e)/(1-e)]} \cdot [\tan(\mathcal{E}/2)]\}. \tag{21}$$

Since we are in the driver program, it is assumed that the orbital elements are all known. One must isolate a value of the time such that equation (17) is satisfied, along with equations (18) through (21). Since no useful analytic solution exists even for equation (20), there is little hope of solving equation (17) analytically. For this reason, a Newton iteration was developed that defined

$$H = \cos(f+\tilde{\omega}) \cdot \cos(\Omega - \phi_o) \cdot \cos\theta_o - \sin(f+\tilde{\omega}) \cdot [\cos I \cdot \sin(\Omega - \phi_o) \cdot \cos\theta_o - \sin I \cdot \sin\theta_o] - R_o/r, \tag{22}$$

in accord with equation (17). To isolate a zero of H, the iterator requires that one also know $\partial H/\partial t$.

$$\partial H/\partial t = -\{\sin(f+\tilde{\omega}) \cdot \cos(\Omega - \phi_o) \cdot \cos\theta_o + \cos(f+\tilde{\omega}) \cdot [\cos I \cdot \sin(\Omega - \phi_o) \cdot \cos\theta_o - \sin I \cdot \sin\theta_o]\} \cdot \sqrt{(p \cdot \mu)}/r^2 - \{\cos(f+\tilde{\omega}) \cdot \sin(\Omega - \phi_o) \cdot \cos\theta_o + \sin(f+\tilde{\omega}) \cdot [\cos I \cdot \cos(\Omega - \phi_o) \cdot \cos\theta_o]\} \cdot \dot{\phi} + (R_o/p) \cdot e \cdot \sin f \cdot \sqrt{(p \cdot \mu)}/r^2, \tag{23}$$

where

$$\partial f/\partial t = \sqrt{[\mu \cdot a \cdot (1-e^2)]}/r^2 = \sqrt{(p \cdot \mu)}/r^2,$$

and

$$\partial/\partial t(1/r) = -(e/p) \cdot \sin f \cdot \sqrt{(p \cdot \mu)}/r^2$$

have been used.

Although the use of equations (22) and (23) can isolate the apparition and occultation of the target, it is convenient to employ approximate calculations to estimate those times as starting values for the Newton iteration. This can be conveniently done if one assumes (here) that $\dot{\phi}$ is zero, but allow f to vary in accord with Keplerian dynamics.

To approximate the apparition and occultation of the target, from equation (13), one defines

$$X_0 = \cos(\Omega - \phi_o) \cdot \cos\theta_o, \tag{24}$$

$$X_1 = [\cos I \cdot \sin(\Omega - \phi_o) \cdot \cos\theta_o - \sin I \cdot \sin\theta_o], \tag{25}$$

so that equation (13) can be written as

$$X_0 \cdot \cos(f+\tilde{\omega}) - X_1 \cdot \sin(f+\tilde{\omega}) - R_o/r = 0. \tag{26}$$

Expanding the double angle formulas in equation (22), and abbreviating

$$X_2 = X_0 \cdot \cos \varpi - X_1 \cdot \sin \varpi, \tag{27}$$

$$X_3 = X_0 \cdot \sin \varpi + X_1 \cdot \cos \varpi, \tag{28}$$

allows equation (22) to be written as

$$X_2 \cdot \cos f - X_3 \cdot \sin f - R_o / r = 0. \tag{29}$$

Inserting r from equation (5) into equation (29) and cross multiplying by the denominator of r gives

$$\cos f \cdot (X_2 - R_o \cdot e/p) - \sin f \cdot X_3 = R_o \cdot e/p. \tag{30}$$

Defining the variable η by

$$\eta = \tan^{-1} [X_3/(X_2 - R_o \cdot e/p)], \tag{31}$$

and ξ as

$$\xi = \cos^{-1} \langle R_o \cdot e/\{p \cdot \sqrt{[X_3^2 + (X_2 - R_o \cdot e/p)^2]}\}\rangle, \tag{32}$$

allows one to write equation (30) as

$$\cos (f + \eta) = \cos \xi. \tag{33}$$

Equation (29) has two solutions, either

$$f = \xi - \eta, \tag{34}$$

or

$$f = 2\pi - (\xi + \eta). \tag{35}$$

Either the solution of equation (34) or the solution of equation (35) will correspond to apparition on a nonrotating Earth; the other will correspond to occultation. Although Gauss' equation assumes that the initial time point occurs at perigee, it is not difficult to introduce a false time reference of zero at apparition by defining this point as t_o and then writing all time values, t, as $t - t_o$. Once the time of apparition has been isolated, this time can be used as a starting point for the equations since the calculation starts at that instant. The second solution for f (which could be the solution of either equation (34) or (35)) then supplies a starting value for the Newton iteration of equation (22) so that the method will converge on occultation rather than apparition.

An orbital table can now be produced by choosing an arbitrary number of points between the f_o (apparition) and the f_f (occultation). Since each f and the orbital elements are known, one can specify all the orbital parameters at these points. Table 1 shows the results of such a procedure for 15 points on a typical orbit. The orbital elements were arbitrarily taken as a = 7,420 km, e = 0.1, ϖ = 25°, I = 35°, and Ω = 33°; ϕ_o was taken as 77° and θ_o as 28.5°, and the Earth's rotation rate, $\dot{\phi}$, was $2\pi/(24 \cdot 3{,}600)$. Apparition occurred at f_o = 8.481° (122.153 seconds past perigee), and occultation occurred at f_f = 50.855° (749.288 seconds past perigee). The remaining entries of table 1 were calculated by the use of equations (5) for r, (10) for ρ, (15) for α, and (11) for β. $\mathcal{E}$ can most easily be obtained from equation (19). The equations that were used to compute the values of the orbital angles ϕ and θ are from standard celestial mechanics

$$\phi = \tan^{-1} \{[\tan (f+\tilde{\omega})\cdot\cos I+\tan \Omega]/[1-\tan (f+\tilde{\omega})\cdot\cos I\cdot\tan \Omega]\} , \tag{36}$$

and

$$\theta = \tan^{-1} [\tan I\cdot\sin (\phi-\Omega)] . \tag{37}$$

It is interesting to determine how large β can become in the field of view. This can be done by requiring that $\partial\beta/\partial f = 0$, and then solving for the corresponding value of f. When this value of f is inserted into equation (11), the maximum value of β will be found. If one uses equation (11), that procedure is algebraically rather messy. It is better to proceed by abbreviating

$$X = \cos (f+\tilde{\omega})\cdot\cos (\Omega-\phi_o)\cdot\cos \theta_o - \sin (f+\tilde{\omega})\cdot[\cos I\cdot\sin (\Omega-\phi_o)\cdot\cos \theta_o - \sin I\cdot\sin \theta_o] , \tag{38}$$

so that equations (10) and (11) can be written, respectively as

$$\rho = \sqrt{(r^2+R_o^2-2\cdot r\cdot R_o\cdot X)} , \tag{39}$$

$$\sin \beta = (r\cdot X-R_o)/\rho = (r\cdot X-R_o)/\sqrt{(r^2+R_o^2-2\cdot r\cdot R_o\cdot X)} , \tag{40}$$

which immediately yields

$$\tan \beta = (r\cdot X-R_o)/[r\cdot\sqrt{(1-X^2)}] . \tag{41}$$

If we now take the derivative of (41), equate $\partial\beta/\partial f$ to zero and clear, we find that the requirement for β to be a maximum is

$$R_o\cdot(1-X^2)\cdot(\partial r/\partial f)+r\cdot[r-R_o\cdot X]\cdot(\partial X/\partial f) = 0 . \tag{42}$$

From equation (15), we have

$$\partial r/\partial f = r^2 e\cdot\sin f/p, \tag{43}$$

while equation (34) gives

$$\partial X/\partial f = -\sin (f+\tilde{\omega})\cdot\cos (\Omega-\phi_o)\cdot\cos \theta_o - \cos (f+\tilde{\omega})\cdot[\cos I\cdot\sin (\Omega-\phi_o)\cdot\cos \theta_o - \sin I\cdot\sin \theta_o] . \tag{44}$$

Taking (34)$\cdot\cos (f+\tilde{\omega})$–(44)$\cdot\sin (f+\tilde{\omega})$ gives an equation which can be solved for $\partial X/\partial f$ as

$$\partial X/\partial f = X\cdot\cot (f+\tilde{\omega})-\cos (\Omega-\phi_o)\cdot\cos \theta_o\cdot\csc (f+\tilde{\omega}) . \tag{45}$$

Inserting equations (15), (43), and (45) into (42) gives, finally,

$$X^2\cdot[e\cdot\cos \tilde{\omega}+\cos (f+\tilde{\omega})]-X\cdot[\cos (\Omega-\phi_o)\cdot\cos \theta_o\cdot(1+e\cdot\cos f)+(p/R_o)\cdot\cos (f+\tilde{\omega})] + [(p/R_o)\cdot\cos (\Omega-\phi_o)\cdot\cos \theta_o + e\cdot\sin f\cdot\sin (f+\tilde{\omega})] = 0 . \tag{46}$$

Equation (46) is cubic in the trigonometric functions of f. Even if one substitutes $\sin^2 f \rightarrow 1-\cos^2 f$, the residual $\sin f$ terms will require an additional squaring operation, so the best that we can hope for is a sixth order equation in $\cos f$. This virtually precludes an analytical solution for f, but numerical iteration works very well. For a numerical iteration, it is easier to use equation (42) (along with equations (38), (43), and (44)) rather than equation (46) (along with equation (38)). Once f has been isolated, the maximum value of β can be obtained from equation (41).

A far easier question is to determine whether or not $\beta = \pi/2$ in the viewing field. From equation (41), β can be $\pi/2$ only if $X = \pm 1$. Using equation (38), this requirement implies that

$$\cos(f+\tilde{\omega})\cdot\cos\theta_o\cdot\cos(\Omega-\phi_o) - \sin(f+\tilde{\omega})\cdot[\cos\theta_o\cdot\cos I\cdot\sin(\Omega-\phi_o) - \sin I\cdot\sin\theta_o] = \pm 1 . \tag{47}$$

Setting

$$\begin{aligned}\tan\lambda &= [\cos\theta_o\cdot\cos I\cdot\sin(\Omega-\phi_o) - \sin I\cdot\sin\theta_o]/[\cos\theta_o\cdot\cos(\Omega-\phi_o)] \\ &= \cos I\cdot\tan(\Omega-\phi_o) - \sin I\cdot\tan\theta_o\cdot\sec(\Omega-\phi_o) ,\end{aligned} \tag{48}$$

then, using spherical trigonometry,

$$\tan\lambda = \cos I\cdot\tan(\Omega-\phi_o) + \sin I\cdot\tan I\cdot\tan(\Omega-\phi_o) = \tan(\Omega-\phi_o)\cdot\sec I . \tag{49}$$

Equation (47) can now be written as

$$\cos(f+\tilde{\omega})\cdot\cos\lambda - \sin(f+\tilde{\omega})\cdot\sin\lambda = \pm 1 ,$$

or

$$\cos(f+\tilde{\omega}+\lambda) = \pm 1 . \tag{50}$$

Thus, β will achieve an angle of $\pi/2$ (or $3\pi/2$) if either of the true anomalies

$$f = -\tan^{-1}[\tan(\Omega-\phi_o)\cdot\sec I] - \tilde{\omega} , \tag{51}$$

or

$$f = \pi - \tan^{-1}[\tan(\Omega-\phi_o)\cdot\sec I] - \tilde{\omega} , \tag{52}$$

fall between apparition and occultation.

From equation (14), an additional datum is available. It is interesting to determine the value of the true anomaly, f, which results in $\alpha = 0$. For this condition, equation (14) requires that

$$\sin(\Omega-\phi_o) + \tan(f+\tilde{\omega})\cdot\cos I\cdot\cos(\Omega-\phi_o) = 0 ,$$

or

$$\tan(f+\tilde{\omega}) = -\sec I\cdot\tan(\Omega-\phi_o) . \tag{53}$$

If ϕ is now taken to be the longitude of the satellite (as opposed ϕ_o, the longitude of the tracking station) a standard equation from celestial mechanics gives

$$\tan(f+\tilde{\omega}) = -\sec I\cdot\tan(\Omega-\phi) . \tag{54}$$

For equations (52) and (53) to hold simultaneously, one must then have $\phi = \phi_o$ as the condition for $\alpha = 0$. At this time

$$f = -\{\tan^{-1}[\sec I \cdot \tan(\Omega - \phi_o)] + \tilde{\omega}\} . \tag{55}$$

If the satellite was observed exactly at $\alpha = 0$ (and the associated time), it would be possible to use this datum to eliminate one orbital element in terms of the remaining four. However, nonlinearities probably make such a procedure more trouble than it is worth.

THE SOLUTION OF THE ANGLES-ONLY PROBLEM

In order to obtain the data in table 1, it was assumed that the orbital elements were known, but that assumption was used only to simulate data that could be recorded by an observer with full knowledge of the orbit. A naive observer who is located at a radar station and restricted to measuring only elevation and azimuth angles (as well as the time of the observations) could record only the sort of information shown in table 2. The "angles only" problem can now be stated quantitatively as follows: given the five sets of readings in table 2, isolate the five orbital elements, a, e, $\tilde{\omega}$, I, and Ω that could have produced those readings. It will be shown that all of the data of that table are not needed. Indeed, one can isolate the required orbital elements from the five reading of either α and the time or β and the time. But listing α and β and the time is redundant. In practice, since all of the reading will be noisy, it would be prudent to solve the problem twice, with one solution involving α and the other solution involving β and then to compare the two answers. Although only five readings can be used to produce the results, these five readings are mathematically precise and thus are "safe." In actual field work, one would probably take many more readings and compute the elements of the orbits from many sets of five, then statistically validate the final answers. These considerations will not be pursued in the present report.

To begin, assume any convenient set of values for a, e, $\tilde{\omega}$, I, and Ω. It is required only that they be reasonable. In most cases, some guess can be made about the semi-major axis (for a satellite, require that $a \cdot (1+e) > R_o$), the eccentricity (say $0 \le e < 1$) and the inclination (if the observation station is at $\theta_o, I \ge \theta_o$).

One could proceed in the calculations in the following way. Once a set of orbital elements have been assumed, one can obtain a value of $\mathcal{E}$ from equation (20) for a given time. Equation (21) then produces the corresponding value of f. Once f has been calculated, obtain ρ from equation (10) and α from equation (15). Equation (11) then gives the value of β. If the guesses about the orbital elements are exact, then the recorded tabular values of α and β will agree with the calculated values of these variables and the problem is solved.

Another procedure was found to be more convenient, however. Given a starting set of guesses (a, e, $\tilde{\omega}$, I, Ω), first calculate X_0, ..., X_3 from equations (26) through (28). Additionally, calculate the following convenient quantities:

$$X_4 = \sin(\Omega - \phi_o) , \tag{56}$$

$$X_5 = \cos I \cdot \cos(\Omega - \phi_o) , \tag{57}$$

$$X_6 = X_4 \cdot \cos\tilde{\omega} + X_5 \cdot \sin\tilde{\omega} , \tag{58}$$

$$X_7 = X_4 \cdot \sin \tilde{\omega} - X_5 \cdot \cos \tilde{\omega} \,, \tag{59}$$

$$X_8 = X_6 \cdot \tan \beta - X_2 \cdot \sin \alpha \,, \tag{60}$$

$$X_9 = X_7 \cdot \tan \beta - X_3 \cdot \sin \alpha \,, \tag{61}$$

$$X_{10} = X_8 \cdot (R_o/p) \cdot \sin \alpha \cdot e \,, \tag{62}$$

$$X_{11} = \sqrt{(X_9^2 + X_{10}^2)} \,, \tag{63}$$

where the tabulated values of α are utilized. In the general case [sin (β) $\neq$ 0], obtain ξ from:

$$\xi = \tan^{-1} (X_{10}/X_9) \,, \tag{64}$$

and η from

$$\eta = \sin^{-1} [(R_o/p) \cdot \sin \alpha / X_{11}] \,. \tag{65}$$

The value of f which corresponds to the given β must be either

$$f = \xi - \eta \,, \tag{66}$$

or

$$f = 2\pi - (\xi + \eta) \,. \tag{67}$$

(The decision of which f to use requires some programming logic, governed by continuity.) Once the value of f is obtained, from the inverse of equation (21),

$$\mathcal{E} = 2 \cdot \tan^{-1} \{ \sqrt{[(1-e)/(1+e)] \cdot \tan (f/2)]} \} \,, \tag{68}$$

with equation (20) then supply the value of time which corresponds to the chosen set of orbital elements at the isolated value of true anomaly.

Unless the orbital elements were guessed precisely, the time which has been computed will not agree with the time which was recorded as a corresponding value to each given α, β, or the (α,β) pair. What is needed is an iteration scheme which will modify the values of the orbital elements in a systematic manner until the calculated times agree with the actual times recorded at the observation site (or the times given by the driver program). Notationally, let $T1_o, \ldots, T5_o$ be the "true" times and let $T_1, \ldots, T_5$ be the times which are calculated from equation (16). Denote U_i as

$$U_i = T_i(a, e, \tilde{\omega}, \Omega, I) - T_{i_o} \quad (i=0, \ldots, 4), \tag{69}$$

and let an "i" subscript denote corresponding time-varying quantities (f, $\mathcal{E}$, etc.).

Suppose that the orbital elements (a, e, $\tilde{\omega}$, I, Ω) differ from the true values of the orbital elements by amounts Δa, Δe, $\Delta\tilde{\omega}$, ΔI, and $\Delta\Omega$ (respectively). If these deviations were known precisely, one would have

$$U_i(a+\Delta a,\ e+\Delta e,\ \tilde{\omega}+\Delta\tilde{\omega},\ \Omega+\Delta\Omega,\ I+\Delta I) = 0. \tag{70}$$

The obvious next step is to expand the U_i set via Taylor's theory to obtain a Newton iterator:

$$U_i(a+\Delta a,\ e+\Delta e,\ I+\Delta I,\ \tilde{\varpi}+\Delta\tilde{\varpi},\ \Omega+\Delta\Omega) = U_i(a,e,I,\tilde{\varpi},\Omega)+(\partial U_i/\partial a)\cdot\Delta a+(\partial U_i/\partial e)$$

$$\cdot\Delta e+(\partial U_i/\partial I)\cdot\Delta I+(\partial U_i/\partial\tilde{\varpi})\cdot\Delta\tilde{\varpi}+(\partial U_i/\partial\Omega)\cdot\Delta\Omega+\ldots = 0, \tag{71}$$

where (i = 0, ..., 4). In matrix notation, with y_j as any member of the ordered set of orbital elements deviations (Δa, Δe, ΔI, $\Delta\tilde{\varpi}$, $\Delta\Omega$), and x_j and member of the ordered set of orbital elements (a, e, I, $\tilde{\varpi}$, Ω), equation (71) becomes:

$$(U_i)^T = (\partial U_i/\partial x_j)\cdot(y_j)^T \quad (i = 0, \ldots, 4), (j = 0, \ldots, 4). \tag{72}$$

Unless $|(\partial U_i/\partial x_j)| = 0$, one can solve for

$$(y_j)^T = (\partial U_i/\partial x_j)^{-1}\cdot(U_i)^T. \tag{73}$$

The values of a, e, I, ϖ, and Ω can now be modified by the substitutions

$$\left.\begin{array}{l} a \rightarrow a+\Delta a \\ e \rightarrow e+\Delta e \\ I \rightarrow I+\Delta I \\ \varpi \rightarrow \varpi+\Delta\varpi \\ \Omega \rightarrow \Omega+\Delta\Omega \end{array}\right\} \tag{74}$$

If the new values of the orbital elements do not drive all of the U_i values to zero, the process can be continued until convergence is achieved.

One still needs to obtain the derivatives of each of the U_i with respect to each of the orbital elements, i.e., the matrix elements of $\partial U_i/\partial x_j$. Temporarily suppressing the "i" subscript, it is convenient to proceed through a "chain calculation" as:

$$\partial U/\partial a = (3/2)\cdot\sqrt{(a/\mu)}\cdot(\mathcal{E}-e\cdot\sin\mathcal{E})+\sqrt{(a^3/\mu)}(1-e\cdot\cos\mathcal{E})\cdot\partial\mathcal{E}/\partial a$$

$$=\sqrt{(a/\mu)}\cdot[(3/2)\cdot(\mathcal{E}-e\cdot\sin\mathcal{E})+a\cdot(1-e\cdot\cos\mathcal{E})\cdot\partial\mathcal{E}/\partial a], \tag{75}$$

$$\partial U/\partial e = \sqrt{(a^3/\mu)}\cdot[(1-e\cdot\cos\mathcal{E})\cdot\partial\mathcal{E}/\partial e], \tag{76}$$

$$\partial U/\partial\tilde{\varpi} = \sqrt{(a^3/\mu)}\cdot[(1-e\cdot\cos\mathcal{E})\cdot\partial\mathcal{E}/\partial\tilde{\varpi}], \tag{77}$$

$$\partial U/\partial\Omega = \sqrt{(a^3/\mu)}\cdot[(1-e\cdot\cos\mathcal{E})\cdot\partial\mathcal{E}/\partial\Omega], \tag{78}$$

$$\partial U/\partial I = \sqrt{(a^3/\mu)}\cdot[(1-e\cdot\cos\mathcal{E})\cdot\partial\mathcal{E}/\partial I]. \tag{79}$$

From equation (68):

$$\partial \mathcal{E}/\partial a = (r/a)\cdot\partial f/\partial a \ , \tag{80}$$

$$\partial \mathcal{E}/\partial e = [r/\sqrt{(1-e^2)}\,][(\partial f/\partial e)/a - \sin f/p], \tag{81}$$

$$\partial \mathcal{E}/\partial \tilde{\omega} = (r/a)\cdot\partial f/\partial \tilde{\omega}, \tag{82}$$

$$\partial \mathcal{E}/\partial \Omega = (r/a)\cdot\partial f/\partial \Omega, \tag{83}$$

$$\partial \mathcal{E}/\partial I = (r/a)\cdot\partial f/\partial I. \tag{84}$$

One will also need the derivatives of ξ from equation (64) and η from equation (65):

$$\frac{\partial\eta}{\partial a} = \frac{-[R_o\cdot\sin\alpha]\cdot(a\cdot\partial X_{11}/\partial a + X_{11})}{[a\cdot X_{11}\cdot\sqrt{(a^2\cdot(1-e^2)^2\cdot X_{11}^2 - [R_o\cdot\sin(\alpha)]^2)}]} \ , \tag{85}$$

$$\frac{\partial\eta}{\partial e} = \frac{\{[R_o\cdot\sin\alpha]/[X_{11}\cdot\cdot(1-e^2)]\}\cdot\{2\cdot e\cdot X_{11} - (1-e^2)\cdot\partial X_{11}/\partial e]}{\sqrt{(a^2\cdot(1-e^2)^2\cdot X_{11}^2 - [R_o\cdot\sin(\alpha)]^2)]\}}} \ , \tag{86}$$

$$\partial\eta/\partial x = ([R_o\cdot\sin\alpha]/X_{11})\cdot\partial X_{11}/\partial x)/\sqrt{(a^2\cdot(1-e^2)^2\cdot X_{11}{}^2 - [R_o\cdot\sin\alpha]^2)}\,], \tag{87}$$

where $x = I$, ϖ, or Ω. For the derivatives of ξ, one has

$$\begin{aligned}\partial\xi/\partial \mathrm{x} &= [\mathrm{X}_9\cdot(\partial\mathrm{X}_{10}/\partial x) - \mathrm{X}_{10}\cdot(\partial\mathrm{X}_9/\partial x)]/[\mathrm{X}_9{}^2 + \mathrm{X}_{10}{}^2] \\ &= [\mathrm{X}_9\cdot(\partial\mathrm{X}_{10}/\partial x) - \mathrm{X}_{10}\cdot(\partial\mathrm{X}_9/\partial x)]/\mathrm{X}_{11}{}^2 \ ,\end{aligned} \tag{88}$$

(for x any orbital element).

Using the given expression of f, equation (66), one is now in a position to calculate the derivatives of f from, say:

$$\partial f/\partial x = \partial\xi/\partial x - \partial\eta/\partial x. \tag{89}$$

The next task is to calculate the derivatives for X_0, ..., X_{11}. These follow (with x as a generic orbital element):

$$\partial X_0/\partial\Omega = -\sin(\Omega - \phi_o)\cdot\cos\theta_o \ , \tag{90}$$

$$\partial X_0/\partial x = 0\ , \ (x \neq \Omega), \tag{91}$$

$$\partial X_1/\partial I = -\sin I\cdot\sin(\Omega - \phi_o)\cdot\cos\theta_o - \cos I\cdot\sin\theta_o \ , \tag{92}$$

$$\partial X_1/\partial\Omega = \cos I\cdot\cos(\Omega - \phi_o)\cdot\cos\theta_o \ , \tag{93}$$

$$\partial X_1/\partial x = 0, \ (x = a, e, \text{ or } \tilde{\varpi}), \tag{94}$$

$$\partial X_2/\partial x = (\partial X_0/\partial x)\cdot\cos\tilde{\varpi} - (\partial X_1/\partial x)\cdot\sin\tilde{\varpi}, \ (x \neq \tilde{\varpi}), \tag{95}$$

$$\partial X_2/\partial\tilde{\varpi} = (\partial X_0/\partial\varpi)\cdot\cos\tilde{\varpi} - (\partial X_1/\partial\tilde{\varpi})\cdot\sin\tilde{\varpi} - X_0\cdot\sin\tilde{\varpi} - X_1\cdot\cos\tilde{\varpi}\ , \tag{96}$$

$$\partial X_3/\partial x = (\partial X_0/\partial x)\cdot\sin\tilde{\varpi} - (\partial X_1/\partial x)\cdot\cos\tilde{\varpi}, \ (x \neq \tilde{\varpi}), \tag{97}$$

$$\partial X_3/\partial\tilde{\varpi} = (\partial X_0/\partial\tilde{\varpi})\cdot\sin\tilde{\varpi} + (\partial X_1/\partial\tilde{\varpi})\cdot\cos\tilde{\varpi} + X_0\cdot\cos\tilde{\varpi} - X_1\cdot\sin\tilde{\varpi}, \tag{98}$$

$$\partial X_4/\partial\Omega = \cos(\Omega - \phi_0)\ , \tag{99}$$

$$\partial X_4/\partial x = 0, \ (x \neq \Omega), \tag{100}$$

$$\partial X_5/\partial I = -\sin I\cdot\cos(\Omega - \phi_0)\ , \tag{101}$$

$$\partial X_5/\partial\Omega = -\cos I\cdot\sin(\Omega - \phi_0)\ , \tag{102}$$

$$\partial X_5/\partial x = 0 \quad (x \neq I, x \neq \Omega), \tag{103}$$

$$\partial X_6/\partial x = (\partial X_4/\partial x)\cdot\cos\tilde{\varpi} + (\partial X_5/\partial x)\cdot\sin\tilde{\varpi}, \ (x \neq \tilde{\varpi}), \tag{104}$$

$$\partial X_6/\partial\tilde{\varpi} = (\partial X_4/\partial\tilde{\varpi})\cdot\cos\tilde{\varpi} + (\partial X_5/\partial\tilde{\varpi})\cdot\sin\tilde{\varpi} - X_4\cdot\sin\tilde{\varpi} + X_5\cdot\cos\tilde{\varpi}\ , \tag{105}$$

$$\partial X_7/\partial x = (\partial X_4/\partial x)\cdot\sin\tilde{\varpi} - (\partial X_5/\partial x)\cdot\cos\tilde{\varpi}, \ (x \neq \tilde{\varpi}), \tag{106}$$

$$\partial X_7/\partial\tilde{\varpi} = (\partial X_4/\partial\tilde{\varpi})\cdot\sin\tilde{\varpi} - (\partial X_5/\partial\tilde{\varpi})\cdot\cos\tilde{\varpi} + X_4\cdot\cos\tilde{\varpi} + X_5\cdot\sin\tilde{\varpi}\ , \tag{107}$$

$$\partial X_8/\partial x = (\partial X_6/\partial x)\cdot\tan\beta - (\partial X_2/\partial x)\cdot\sin\alpha\ , \tag{108}$$

$$\partial X_9/\partial x = (\partial X_7/\partial x)\cdot\tan\beta - (\partial X_3/\partial x)\cdot\sin\alpha, \tag{109}$$

$$\partial X_{10}/\partial x = \partial X_8/\partial x, \ (x \neq e), \tag{110}$$

$$\partial X_{10}/\partial e = \partial X_8/\partial e + R_o\cdot\sin\alpha\ , \tag{111}$$

$$\partial X_{11}/\partial x = [X_9\cdot(\partial X_9/\partial x) + X_{10}\cdot(\partial X_{10}/\partial x)]/X_{11}. \tag{112}$$

If the program utilizes both the α and β table, the development is now complete. Once the orbital elements have been chosen, we can obtain the derivatives of $X_0, \ldots, X_{11}$ with respect to $(a, e, \tilde{\varpi}, I, \Omega)$ each orbital element from equations (90) through (112). Given these derivatives, use equations (85) through (88) to obtain the derivatives of η and ξ with respect to the orbital elements, and subsequently, the derivatives of f with respect to the same variables. That allows a computation of the derivatives of ε (equations (80) though (84)) and finally the derivatives of U from equations (75) through (79). This yields the derivatives of a five-dimensional vector, say $(\partial U_0/\partial x)$. The process is repeated for each of the other data sets (α_i, β_i, T_i), to construct the matrix shown in equation (72). Matrix manipulation then yields the corrections to the orbital elements from equation(s) (74).

If the information on the azimuth, α (or β), is not used, the situation is more complicated since α must be taken as another variable and equations (60), (61) and (62) must account for the fact that α is defined in terms of the assumed orbital elements. From equation (11), one can use equations (58) and (59) to write

$$\sin\alpha = r\cdot(X_6\cdot\cos f - X_7\cdot\sin f)/(\rho\cdot\cos\beta), \tag{113}$$

while equation (10) becomes

$$\rho = \sqrt{[r^2+R_o^2-2\cdot r\cdot R_o\cdot(X_2\cdot\cos f - X_3\cdot\sin f)]}\,. \tag{114}$$

Equations (107) through (111) would be modified to read:

$$\partial X_8/\partial x = (\partial X_6/\partial x)\cdot\tan\beta - (\partial X_2/\partial x)\cdot\sin\alpha - X_2\cdot\cos\alpha\cdot\partial\alpha/\partial x, \tag{115}$$

$$\partial X_9/\partial x = (\partial X_7/\partial x)\cdot\tan\beta - (\partial X_3/\partial x)\cdot\sin\alpha - X_3\cdot\cos\alpha\cdot\partial\alpha/\partial x, \tag{116}$$

$$\partial X_{10}/\partial x = \partial X_8/\partial x + R_o/p)\cdot\cos\alpha\cdot\partial\alpha/\partial x \ \ (x \neq e), \tag{117}$$

$$\partial X_{10}/\partial e = \partial X_8/\partial e + R_o\cdot\sin\alpha + (R_o/p)\cdot\cos\alpha\cdot\partial\alpha/\partial x \ \ (x = e). \tag{118}$$

From equation (114) comes

$$\partial\rho/\partial x = \{(r-R_o)\cdot\partial r/\partial x - r\cdot R_o\cdot[\cos f\cdot(\partial X_2/\partial x - X_3\cdot\partial f/\partial x)$$

$$-\sin f\cdot(\partial X_3/\partial x + X_2\cdot\partial f/\partial x]\}/\rho\,, \tag{119}$$

which is needed to evaluate $\partial\alpha/\partial x$; from equation (113) one obtains both

$$\cos\alpha\cdot\partial\alpha/\partial x = \{\partial r/\partial x\cdot(X_6\cdot\cos f - X_7\cdot\sin f) + r\cdot[(\partial X_6/\partial x - X_7\cdot\partial f/\partial x)\cdot\cos f$$

$$-(X_6\cdot\partial f/\partial x + \partial X_7/\partial x)\cdot\sin f]\}/(\rho\cdot\cos\beta) - r\cdot(X_6\cdot\cos f - X_7\cdot\sin f)\cdot\partial\rho/\partial t/(\rho^2\cdot\cos\beta)\,, \tag{120}$$

and

$$\cos\alpha = \sqrt{[\rho^2\cdot\cos^2\beta - r^2\cdot(X_6\cdot\cos f - X_7\cdot\sin f)^2]}\,/(\rho\cdot\cos\beta)\,. \tag{121}$$

In order to take the derivatives of α with respect to each orbital element, one would also need the derivatives of r with respect to the same variables. Using equation (19), one finds

$$\partial r/\partial a = 1 - e\cdot\cos\mathcal{E} + a\cdot e\cdot\sin\mathcal{E}\cdot\partial\mathcal{E}/\partial a, \tag{122}$$

$$\partial r/\partial e = a\cdot\cos\mathcal{E} + a\cdot e\cdot\sin\mathcal{E}\cdot\partial\mathcal{E}/\partial e\,. \tag{123}$$

Using equations (115) through (123) allows the measured values for α to be ignored, but ignoring the measured values is paid for with the need for increased computation. Similar comments are applicable to the case where tabular values of α are assumed to be known but β is treated as a computable quantity. In the numerical results that are presented in the next section, tabular values of both α and β were assumed to be known.

NUMERICAL RESULTS

The above theory was programmed in COMMON LISP on a Symbolics 6370. That platform allows for rapid prototyping and ease of equation development. The entire set of equations performed quite well; the Newton iteration scheme can converge a correct set of orbital elements from rather bad guesses.

There has been an implicit assumption that there is a unique set of orbital elements which will produce the given table of observations. Although no attempt was made to establish uniqueness, one case was inadvertently found where the solution was not mathematically unique, but the extraneous solution did not correspond to a physically realistic orbit. Specifically, for the inputs of α, β, and time from a two-dimensional suborbital case (with true $a = 6{,}018.739$, $e = 0.724$, $\tilde{\omega} = 225$, $I = 0$, and $\delta\Omega = 0$), started from a rather precise initial set of guesses that assumed $a = 6{,}020$, $e = 0.5$, $\tilde{\omega} = 350$, I=0, $\delta\Omega = 0$. The convergence criteria was met with $a = 9{,}666.27930$, $e = -0.17980$, $\tilde{\omega} = -25.54499$, $I = 0$, and $\delta\Omega = 0$. This is not a physically realistic orbit ($e < 0$, $a > 0$), however, and is easily eliminated by requiring that $e \geq 0$. Additional constraints should be added to the iteration scheme, i.e., require that $a \cdot (1+e) \geq R_o$, $-\pi/2 \leq I \leq \pi/2$, and $0 \leq \delta\Omega \leq \pi$. In the case of tracking a comet or an asteroid (which are usually elliptical with respect to the Sun but hyperbolic with respect to the Earth), alternative constraints obviously would have to be used.

In regard to two-dimensional iterations, in general, note that the convergence scheme is valid for those cases provided that the reduced dimension of the iteration space is accounted for. A three-dimensional iteration, faced with a two-dimensional problem, will encounter a singular matrix unless precautions are taken. Furthermore, when a two-dimensional problem is undertaken, one must take $\alpha = 3\pi/2$ due to the choice of the coordinate system.

During the computation procedure, the equation for η, equation (31), was particularly treacherous in that it is rather easy for $(R_o/p) \cdot \sin\,\alpha/X11$ to underflow or overflow from the range $-1 \leq \sin^{-1} \leq 1$ and produce complex values for α. The LISP language easily handled the complex values, but it balked at attempting to double-float the results. This was handled by the crude, though effective, artifice of simply setting α to either $3\pi/2$ or $\pi/2$ depending upon the direction in which the insult occurred. As the iteration scheme approached more accurate values of the orbital elements, the problem disappeared. Iteration to isolate a satellite orbit tends to monotone convergence, but isolation of very short range missiles (~200 km) becomes extremely difficult since the trajectories have eccentricities that approach unity. For those cases, the Newton iteration, as presented here, is virtually inapplicable.

Another difficulty, often present in Newtonian iteration schemes, was that the corrections for Δa, Δe, $\Delta\tilde{\omega}$, ΔI and $\Delta\delta\Omega$ were often so large that the iteration scheme began to wander aimlessly. Two methods were used to circumvent this problem. One option measured the magnitude of the errors and adjusted a "creep factor" according to the size of the errors. Another scheme placed absolute bounds on the allowable changes that could be made by any variable in a given iteration. This report was written to demonstrate a new approach to the angles-only problem, so no attempt was made to develop an optimal iteration scheme. Since the angles-only problem is important in the real world, it is expected that specific iteration schemes to optimize convergence may be forthcoming in the future.

Two illustrative examples are given here to demonstrate the use of the equations given earlier. As mentioned previously, the first of these is a rather standard satellite orbit which is numerically described in Table 1; the fact that 16 sets of data are recorded is purely arbitrary. A subset of the data displayed in Table 1 is shown in Table 2. This is the entire set of data sent to the iteration scheme. As explained, the numerical examples used the values of α and β along the time of the observations, so that neither α nor

β were calculated from the other as a part of the study. The iteration scheme converged the proper elements from the input guesses in 26 iterations (Table 3). Note that the figure of merit for the convergence (last column, Table 3) is monotone decreasing. The error calculation is defined as $\sqrt{U_0^2+U_1^2+U_2^2+U_3^2+U_4^2}$.

Table 4 demonstrates the same data as Table 1, but this time for a suborbital vehicle. The case was deliberately taken as equatorial so that a two-dimensional iteration could be demonstrated. Since only three orbital elements (a, e, $\tilde{\omega}$) need to be isolated, one can make-do with only three readings. For a two-dimensional case, if the tracking station is located in the plane of the orbit, the azimuth angle, α, is meaningless. Thus, only three readings are needed. Again, the end points were $\beta = 0$ or $\beta = \pi$ were included, but this is arbitrary. Table 4 shows the results of the iteration scheme from very poor initial guesses; indeed, the guesses are about as bad as they can meaningfully be. The iteration scheme does not display monotone convergence and appears to wander aimless for 61 of the 107 iterations that were needed to converge the case.

An attempt was made to further stress the iteration scheme by requesting an isolation of the 200-km range suborbital planar case, but convergence was never attained. Certainly, the minimal Newton iteration presented here can be strengthened by any of several known methods.

CONCLUSIONS

The preceding theory has derived another solution to the "angles-only" problem. This method, possibly used in conjunction with the alternative methods covered in reference 1, has the potential to decrease the amount of hardware that is currently required for rendezvous and missile tracking.

The numerical work that is presented demonstrates that the theory yields a practical scheme that actually solves the problem. Unfortunately, it also indicates that additional research is required to extend numerical convergence to eccentricities which are very close to unity.

A planned future paper will relax the requirement of an Earth-fixed observer and treat the case of an observer in an orbiting satellite.

Table 1. First example.

SEMI-MAJOR AXIS (A)=7420. KM, ECC=.10000 INC(I)=35.000 DEG., ARG. PERIGEE(ω)=25.000 DEG., LONG. ASC. NODE(Ω)=33.000 DEG.
APPARITION OCCURS AT f=8.481 deg. [T=122.154 sec.] AND OCCULTATION OCCURS AT f=50.309 deg. [T=740.883 sec.]

TRUE ANOMOLY, DEG.	RADIUS, KM.	RANGE, KM.	ECC. ANOMOLY, DEG.	TIME, SEC.
0 f=8.48133608	r=6684.64603564	ρ=2001.40179057	ℰ=7.67420293	T=122.15382475
1 f=11.33937028	r=6689.87187027	ρ=1702.55541086	ℰ=10.26292799	T=163.40242366
2 f=14.19230929	r=6696.58128091	ρ=1406.84597518	ℰ=12.84934444	T=204.65102256
3 f=17.03891388	r=6704.75468903	ρ=1117.35905908	ℰ=15.43288791	T=245.89962147
4 f=19.87797554	r=6714.36839048	ρ=841.02426446	ℰ=18.01300611	T=287.14822037
5 f=22.70832198	r=6725.39470844	ρ=596.64903616	ℰ=20.58916104	T=328.39681928
6 f=25.52882210	r=6737.80216907	ρ=440.75876006	ℰ=23.16083097	T=369.64541818
7 f=28.33839058	r=6751.55569755	ρ=470.56098382	ℰ=25.72751237	T=410.89401708
8 f=31.13599196	r=6766.61683174	ρ=660.49278407	ℰ=28.28872158	T=452.14261599
9 f=33.92064421	r=6782.94395090	ρ=914.90251289	ℰ=30.84399635	T=493.39121489
10 f=36.69142178	r=6800.49251644	ρ=1192.33738686	ℰ=33.39289717	T=534.63981380
11 f=39.44745811	r=6819.21532193	ρ=1479.03113160	ℰ=35.93500846	T=575.88841270
12 f=42.18794762	r=6839.06274951	ρ=1769.64478692	ℰ=38.46993949	T=617.13701161
13 f=44.91214716	r=6859.98302993	ρ=2061.70019529	ℰ=40.99732514	T=658.38561051
14 f=47.61937693	r=6881.92250357	ρ=2353.86307735	ℰ=43.51682652	T=699.63420942
15 f=50.30902095	r=6904.82588003	ρ=2645.32323647	ℰ=46.02813134	T=740.88280832

AZIMUTH, DEG.	ELEVATION, DEG.	LONGITUDE, DEG.	LATITUDE, DEG.	LONGITUE (RADAR) DEG.
0 α=-58.15003437	β=-.00000113	Φ=61.44889048	Θ=18.44669380	Φs=77.00000000
1 α=-57.00645790	β=3.10616670	Φ=64.07291615	Θ=19.86970746	Φs=77.17186916
2 α=-55.34118427	β=6.99699965	Φ=66.73900244	Θ=21.25119328	Φs=77.34373832
3 α=-52.71266671	β=12.25898824	Φ=69.44867889	Θ=22.58706924	Φs=77.51560749
4 α=-48.00953651	β=20.15528607	Φ=72.20310392	Θ=23.87327468	Φs=77.68747665
5 α=-37.56514360	β=33.45800818	Φ=75.00300642	Θ=25.10579627	Φs=77.85934581
6 α=-4.22028134	β=53.59066411	Φ=77.84862649	Θ=26.28069784	Φs=78.03121497
7 α=64.33158348	β=128.72017840	Φ=80.73965749	Θ=27.39415375	Φs=78.20308413
8 α=90.55732702	β=146.32232990	Φ=83.67519231	Θ=28.44248594	Φs=78.37495330
9 α=99.36073597	β=157.36021261	Φ=86.65367667	Θ=29.42220398	Φs=78.54682246
10 α=103.52061532	β=164.18076975	Φ=89.67287325	Θ=30.33004762	Φs=78.71869162
11 α=105.90245379	β=168.89052738	Φ=92.72983999	Θ=31.16303074	Φs=78.89056078
12 α=107.42823026	β=172.45952834	Φ=95.82092612	Θ=31.91848556	Φs=79.06242995
13 α=108.47830717	β=175.35471125	Φ=98.94178883	Θ=32.59410553	Φs=79.23429911
14 α=109.23705338	β=177.82162762	Φ=102.08743269	Θ=33.18798516	Φs=79.40616827
15 α=109.80439856	β=179.99999761	Φ=105.25227275	Θ=33.69865517	Φs=79.57803743

Table 2. Representative data of entire package sent to iterator.

(μ, Φo, Φ̇, Θo, Ro ASSUMED KNOWN)

T=122.15382475	0 α=-58.15003437	β=-.00000113
T=276.83607064	1 α=-49.49765192	β=17.81825854
T=431.51831654	2 α=81.48848503	β=41.79462703
T=586.20056243	3 α=106.34412297	β=169.86594977
T=740.88280832	4 α=109.80439856	β=179.99999761

Table 3. Results of iteration scheme.

ORIGINAL GUESSES: A=7000.00000 km Ecc=.00000 Io=27.00000 deg ω=.00000 deg Ω=.00000 deg

ITERATION #0	A=6997.11928 km	Ecc=.00153	ω=00.70312 deg	Io=27.63809 deg	Ω=359.66414 deg	√[SUM (ERRORS)²]=2009.63630547
ITERATION #1	A=6976.37418 km	Ecc=.01593	ω=04.53330 deg	Io=34.23148 deg	Ω=000.10470 deg	√[SUM (ERRORS)²]=1989.44887953
ITERATION #2	A=6984.14600 km	Ecc=.02578	ω=04.30058 deg	Io=34.72687 deg	Ω=005.38340 deg	√[SUM (ERRORS)²]=1784.30766470
ITERATION #3	A=7001.23236 km	Ecc=.03438	ω=05.26028 deg	Io=35.26693 deg	Ω=009.02111 deg	√[SUM (ERRORS)²]=1605.87646748
ITERATION #4	A=7023.14477 km	Ecc=.04188	ω=06.72678 deg	Io=35.64320 deg	Ω=011.73783 deg	√[SUM (ERRORS)²]=1445.02945689
ITERATION #5	A=7047.51661 km	Ecc=.04841	ω=08.35303 deg	Io=35.86671 deg	Ω=013.91800 deg	√[SUM (ERRORS)²]=1300.14138118
ITERATION #6	A=7072.88632 km	Ecc=.05411	ω=09.97658 deg	Io=35.97578 deg	Ω=015.75416 deg	√[SUM (ERRORS)²]=1169.70535455
ITERATION #7	A=7098.31449 km	Ecc=.05910	ω=11.51927 deg	Io=36.00527 deg	Ω=017.35192 deg	√[SUM (ERRORS)²]=1052.31412419
ITERATION #8	A=7160.51885 km	Ecc=.07010	ω=15.08418 deg	Io=35.94827 deg	Ω=020.90512 deg	√[SUM (ERRORS)²]=0946.67896590
ITERATION #9	A=7212.08151 km	Ecc=.07731	ω=17.84513 deg	Io=35.62406 deg	Ω=023.55637 deg	√[SUM (ERRORS)²]=0708.35808592
ITERATION #10	A=7254.11223 km	Ecc=.08240	ω=19.81318 deg	Io=35.33142 deg	Ω=025.69427 deg	√[SUM (ERRORS)²]=0530.02498390
ITERATION #11	A=7288.44022 km	Ecc=.08623	ω=21.20240 deg	Io=35.13819 deg	Ω=027.41042 deg	√[SUM (ERRORS)²]=0396.60902847
ITERATION #12	A=7316.43888 km	Ecc=.08921	ω=22.19628 deg	Io=35.02999 deg	Ω=028.75806 deg	√[SUM (ERRORS)²]=0296.80609278
ITERATION #13	A=7339.12253 km	Ecc=.09158	ω=22.91851 deg	Io=34.97731 deg	Ω=029.79734 deg	√[SUM (ERRORS)²]=0222.14316318
ITERATION #14	A=7357.31780 km	Ecc=.09347	ω=23.44940 deg	Io=34.95647 deg	Ω=030.58963 deg	√[SUM (ERRORS)²]=0166.27911558
ITERATION #15	A=7371.75187 km	Ecc=.09497	ω=23.84247 deg	Io=34.95223 deg	Ω=031.18950 deg	√[SUM (ERRORS)²]=0124.47395496
ITERATION #16	A=7394.40775 km	Ecc=.09732	ω=24.42709 deg	Io=34.95924 deg	Ω=032.09412 deg	√[SUM (ERRORS)²]=0093.18528045
ITERATION #17	A=7406.78486 km	Ecc=.09861	ω=24.71504 deg	Io=34.97549 deg	Ω=032.54899 deg	√[SUM (ERRORS)²]=0046.30360731
ITERATION #18	A=7413.29293 km	Ecc=.09929	ω=24.85799 deg	Io=34.98695 deg	Ω=032.77596 deg	√[SUM (ERRORS)²]=0023.02310822
ITERATION #19	A=7416.62822 km	Ecc=.09964	ω=24.92917 deg	Io=34.99342 deg	Ω=032.88883 deg	√[SUM (ERRORS)²]=0011.45162358
ITERATION #20	A=7418.31322 km	Ecc=.09982	ω=24.96465 deg	Io=34.99676 deg	Ω=032.94487 deg	√[SUM (ERRORS)²]=0005.69708321
ITERATION #21	A=7419.15825 km	Ecc=.09991	ω=24.98236 deg	Io=34.99843 deg	Ω=032.97267 deg	√[SUM (ERRORS)²]=0002.83453385
ITERATION #22	A=7419.58049 km	Ecc=.09995	ω=24.99119 deg	Io=34.99924 deg	Ω=032.98645 deg	√[SUM (ERRORS)²]=0001.41037319
ITERATION #23	A=7420.00172 km	Ecc=.10000	ω=25.00001 deg	Io=35.00003 deg	Ω=033.00012 deg	√[SUM (ERRORS)²]=0000.70177748
ITERATION #24	A=7420.00025 km	Ecc=.10000	ω=25.00000 deg	Io=35.00000 deg	Ω=033.00000 deg	√[SUM (ERRORS)²]=0000.00353102
ITERATION #25	A=7420.00026 km	Ecc=.10000	ω=25.00000 deg	Io=35.00000 deg	Ω=033.00000 deg	√[SUM (ERRORS)²]=0000.00001940
ITERATION #26	A=7420.00026 km	Ecc=.10000	ω=25.00000 deg	Io=35.00000 deg	Ω=033.00000 deg	√[SUM (ERRORS)²]=0000.00000010

CONVERGED SOLUTION: A=7420.00026 km Ecc=.10000 Io=35.00000 deg ω=25.00000 deg Ω=33.00000 deg
√[SUM (ERRORS)²]=.00000010

Table 4. Second example.

SEMI-MAJOR AXIS (A)=6019. KM, ECC=.72428 INC(I)=.000 DEG., ARG. PERIGEE(ω)=225.000 DEG., LONG. ASC. NODE(Ω)=.000 DEG.
APPARITION OCCURS AT f=154.991 deg. [T=1120.749 sec.] AND OCCULTATION OCCURS AT f=219.128 deg. [T=3932.605 sec.]

TRUE ANOMOLY, DEG.	RADIUS, KM.	RANGE, KM.	ECC. ANOMOLY, DEG.	TIME, SEC.
0 f=154.99056464	r=8327.03917718	ρ=5353.56799812	ℰ=121.97277140	T=1120.74879014
1 f=159.84812332	r=8940.30401130	ρ=5316.69080146	ℰ=132.08228159	T=1308.20587948
2 f=164.13710554	r=9434.24059567	ρ=5301.21356239	ℰ=141.58258033	T=1495.66296882
3 f=168.04340171	r=9818.44038207	ρ=5278.16143365	ℰ=150.64963478	T=1683.12005817
4 f=171.69290062	r=10099.63885244	ρ=5226.61914198	ℰ=159.41388293	T=1870.57714751
5 f=175.17806710	r=10282.40302165	ρ=5131.41085411	ℰ=167.97870649	T=2058.03423685
6 f=178.57269911	r=10369.54893382	ρ=4981.20243176	ℰ=176.43164170	T=2245.49132620
7 f=181.94118337	r=10362.37915844	ρ=4767.10213407	ℰ=184.85198369	T=2432.94841554
8 f=185.34530128	r=10260.78748541	ρ=4481.69994810	ℰ=193.31671190	T=2620.40550488
9 f=188.85039710	r=10063.25051039	ρ=4118.50129078	ℰ=201.90596867	T=2807.86259422
10 f=192.53242283	r=9766.70455542	ρ=3671.85978114	ℰ=210.70915095	T=2995.31968357
11 f=196.48778574	r=9366.28463255	ρ=3137.99737918	ℰ=219.83291254	T=3182.77677291
12 f=200.84937183	r=8854.87299320	ρ=2519.60614320	ℰ=229.41318022	T=3370.23386225
13 f=205.81589808	r=8222.35862730	ρ=1846.25538555	ℰ=239.63526416	T=3557.69095160
14 f=211.71203239	r=7454.43313420	ρ=1281.26945377	ℰ=250.77108362	T=3745.14804094
15 f=219.12843774	r=6530.64834910	ρ=1403.73789664	ℰ=263.25619535	T=3932.60513028

AZIMUTH, DEG.	ELEVATION, DEG.	LONGITUDE, DEG.	LATITUDE, DEG.	LONGITUE (RADAR) DEG.
0 α=-90.00000000	β=.00000600	Φ=19.99056464	Θ=.00000000	Φs=60.00000000
1 α=-90.00000000	β=9.31971328	Φ=24.84812332	Θ=.00000000	Φs=60.78107121
2 α=-90.00000000	β=17.40117751	Φ=29.13710554	Θ=.00000000	Φs=61.56214241
3 α=-90.00000000	β=24.44702159	Φ=33.04340171	Θ=.00000000	Φs=62.34321362
4 α=-90.00000000	β=30.66801485	Φ=36.69290062	Θ=.00000000	Φs=63.12428482
5 α=-90.00000000	β=36.26370188	Φ=40.17806710	Θ=.00000000	Φs=63.90535603
6 α=-90.00000000	β=41.41969757	Φ=43.57269911	Θ=.00000000	Φs=64.68642723
7 α=-90.00000000	β=46.31600507	Φ=46.94118337	Θ=.00000000	Φs=65.46749844
8 α=-90.00000000	β=51.14502534	Φ=50.34530128	Θ=.00000000	Φs=66.24856964
9 α=-90.00000000	β=56.14476239	Φ=53.85039710	Θ=.00000000	Φs=67.02964085
10 α=-90.00000000	β=61.66664772	Φ=57.53242283	Θ=.00000000	Φs=67.81071206
11 α=-90.00000000	β=68.33790782	Φ=61.48778574	Θ=.00000000	Φs=68.59178326
12 α=-90.00000000	β=77.52662540	Φ=65.84937183	Θ=.00000000	Φs=69.37285447
13 α=-90.00000000	β=87.05064752	Φ=70.81589808	Θ=.00000000	Φs=70.15392567
14 α=-90.00000000	β=125.84723511	Φ=76.71203239	Θ=.00000000	Φs=70.93499688
15 α=-90.00000000	β=179.99998791	Φ=84.12843774	Θ=.00000000	Φs=71.71606808

Table 5. Representative data of entire package sent to iterator.

(μ, Φo, Φ̇, Θo, Ro ASSUMED KNOWN)

T=1120.74879014	β=.00000600
T=2526.67696021	β=48.72566711
T=3932.60513028	β=179.99998791

Table 6. Results of iteration scheme.

ORIGINAL GUESSES: A=6378.00000 km Ecc=.00000 Io=.00000 deg ω=.00000 deg Ω=.00000 deg

ITERATION #0:	A=6377.82547 km	Ecc=.01000	ω=246.53286 deg	√[SUM (ERRORS)²]=3342.25372390
ITERATION #1:	A=6377.65093 km	Ecc=.11000	ω=275.73814 deg	√[SUM (ERRORS)²]=1707.90104467
ITERATION #2:	A=6377.47640 km	Ecc=.21000	ω=243.58550 deg	√[SUM (ERRORS)²]=1563.56658065
ITERATION #3:	A=6377.30187 km	Ecc=.31000	ω=214.12491 deg	√[SUM (ERRORS)²]=0967.94274041
ITERATION #4:	A=6377.47640 km	Ecc=.41000	ω=258.37404 deg	√[SUM (ERRORS)²]=1210.41028906
ITERATION #5:	A=6377.30187 km	Ecc=.51000	ω=119.31538 deg	√[SUM (ERRORS)²]=0862.33919423
ITERATION #6:	A=6377.47640 km	Ecc=.41000	ω=161.91792 deg	√[SUM (ERRORS)²]=4237.05560753
ITERATION #7:	A=6377.30187 km	Ecc=.31000	ω=143.56112 deg	√[SUM (ERRORS)²]=3587.49167198
ITERATION #8:	A=6377.47640 km	Ecc=.21000	ω=002.73302 deg	√[SUM (ERRORS)²]=3902.12313582
ITERATION #9:	A=6377.65093 km	Ecc=.11000	ω=102.47098 deg	√[SUM (ERRORS)²]=3642.75359697
ITERATION #10:	A=6377.82547 km	Ecc=.01000	ω=218.38686 deg	√[SUM (ERRORS)²]=3958.24350289
ITERATION #11:	A=6377.65093 km	Ecc=.11000	ω=076.02424 deg	√[SUM (ERRORS)²]=1850.50353792
ITERATION #12:	A=6377.82547 km	Ecc=.01000	ω=060.90440 deg	√[SUM (ERRORS)²]=4175.96923678
ITERATION #13:	A=6377.65093 km	Ecc=.11000	ω=110.29098 deg	√[SUM (ERRORS)²]=5569.74171210
ITERATION #14:	A=6377.82547 km	Ecc=.01000	ω=133.66279 deg	√[SUM (ERRORS)²]=3903.96075415
ITERATION #15:	A=6378.00000 km	Ecc=.01000	ω=110.56111 deg	√[SUM (ERRORS)²]=3621.66570190
ITERATION #16:	A=6378.00000 km	Ecc=.01000	ω=220.70121 deg	√[SUM (ERRORS)²]=3759.25058461
ITERATION #17:	A=6377.82547 km	Ecc=.11000	ω=336.81249 deg	√[SUM (ERRORS)²]=1828.42076689
ITERATION #18:	A=6378.00000 km	Ecc=.01000	ω=056.21260 deg	√[SUM (ERRORS)²]=3104.59657251
ITERATION #19:	A=6377.82547 km	Ecc=.11000	ω=036.52690 deg	√[SUM (ERRORS)²]=4543.88574404
ITERATION #20:	A=6378.00000 km	Ecc=.01000	ω=107.39903 deg	√[SUM (ERRORS)²]=4167.53932133
ITERATION #21:	A=6378.00000 km	Ecc=.01000	ω=230.21842 deg	√[SUM (ERRORS)²]=3783.94734648
ITERATION #22:	A=6377.82547 km	Ecc=.11000	ω=097.93358 deg	√[SUM (ERRORS)²]=1756.34696722
ITERATION #23:	A=6378.00000 km	Ecc=.01000	ω=262.73277 deg	√[SUM (ERRORS)²]=3991.99014902
ITERATION #24:	A=6378.00000 km	Ecc=.01000	ω=037.54403 deg	√[SUM (ERRORS)²]=1758.09479345
ITERATION #25:	A=6378.00000 km	Ecc=.01000	ω=074.54306 deg	√[SUM (ERRORS)²]=4138.43426753
ITERATION #26:	A=6378.00000 km	Ecc=.01000	ω=274.77163 deg	√[SUM (ERRORS)²]=4115.06597338
ITERATION #27:	A=6378.00000 km	Ecc=.01000	ω=047.80174 deg	√[SUM (ERRORS)²]=1855.01818369
ITERATION #28:	A=6378.00000 km	Ecc=.01000	ω=322.64812 deg	√[SUM (ERRORS)²]=4360.17724804
ITERATION #29:	A=6378.00000 km	Ecc=.01000	ω=062.38489 deg	√[SUM (ERRORS)²]=2705.93322260
ITERATION #30:	A=6377.82547 km	Ecc=.11000	ω=116.39484 deg	√[SUM (ERRORS)²]=5573.56804375
ITERATION #31:	A=6378.00000 km	Ecc=.01000	ω=059.07265 deg	√[SUM (ERRORS)²]=3864.88321210
ITERATION #32:	A=6377.82547 km	Ecc=.11000	ω=354.23826 deg	√[SUM (ERRORS)²]=4606.71200290
ITERATION #33:	A=6378.00000 km	Ecc=.01000	ω=326.15469 deg	√[SUM (ERRORS)²]=3371.70846483
ITERATION #34:	A=6378.00000 km	Ecc=.01000	ω=028.42436 deg	√[SUM (ERRORS)²]=2762.87190362
ITERATION #35:	A=6378.00000 km	Ecc=.01000	ω=252.93308 deg	√[SUM (ERRORS)²]=3943.68348011

Table 6. Results of iteration scheme (continued).

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ITERATION  #36: A=6378.00000 km Ecc=.01000 ω=064.59755 deg √[SUM (ERRORS)²]=1715.94468123
ITERATION  #37: A=6377.82547 km Ecc=.11000 ω=125.01488 deg √[SUM (ERRORS)²]=5579.52089429
ITERATION  #38: A=6378.00000 km Ecc=.01000 ω=295.14937 deg √[SUM (ERRORS)²]=3814.22587824
ITERATION  #39: A=6378.00000 km Ecc=.01000 ω=059.07106 deg √[SUM (ERRORS)²]=2112.08775784
ITERATION  #40: A=6377.82547 km Ecc=.11000 ω=354.26307 deg √[SUM (ERRORS)²]=4606.67699080
ITERATION  #41: A=6378.00000 km Ecc=.01000 ω=325.54414 deg √[SUM (ERRORS)²]=3372.17423267
ITERATION  #42: A=6378.00000 km Ecc=.01000 ω=034.50583 deg √[SUM (ERRORS)²]=2752.92595544
ITERATION  #43: A=6378.00000 km Ecc=.01000 ω=132.14604 deg √[SUM (ERRORS)²]=4073.28820556
ITERATION  #44: A=6378.00000 km Ecc=.01000 ω=121.06298 deg √[SUM (ERRORS)²]=3628.40253180
ITERATION  #45: A=6378.00000 km Ecc=.01000 ω=181.18194 deg √[SUM (ERRORS)²]=3687.17028436
ITERATION  #46: A=6377.82547 km Ecc=.11000 ω=094.81280 deg √[SUM (ERRORS)²]=3172.60797744
ITERATION  #47: A=6378.00000 km Ecc=.01000 ω=291.13616 deg √[SUM (ERRORS)²]=4015.96918752
ITERATION  #48: A=6378.00000 km Ecc=.01000 ω=146.17872 deg √[SUM (ERRORS)²]=2053.68258407
ITERATION  #49: A=6378.00000 km Ecc=.01000 ω=343.47329 deg √[SUM (ERRORS)²]=3580.81284395
ITERATION  #50: A=6378.00000 km Ecc=.01000 ω=218.86757 deg √[SUM (ERRORS)²]=3030.67410247
ITERATION  #51: A=6377.82547 km Ecc=.11000 ω=055.29646 deg √[SUM (ERRORS)²]=1845.65847847
ITERATION  #52: A=6378.00000 km Ecc=.21000 ω=048.84146 deg √[SUM (ERRORS)²]=4523.50960091
ITERATION  #53: A=6377.82547 km Ecc=.31000 ω=260.19530 deg √[SUM (ERRORS)²]=4415.62354477
ITERATION  #54: A=6377.65093 km Ecc=.41000 ω=038.99094 deg √[SUM (ERRORS)²]=0964.67445113
ITERATION  #55: A=6377.47640 km Ecc=.31000 ω=211.81071 deg √[SUM (ERRORS)²]=4293.98179749
ITERATION  #56: A=6377.65093 km Ecc=.41000 ω=263.74250 deg √[SUM (ERRORS)²]=1283.33023873
ITERATION  #57: A=6377.47640 km Ecc=.51000 ω=083.08813 deg √[SUM (ERRORS)²]=1011.94406805
ITERATION  #58: A=6377.30187 km Ecc=.61000 ω=312.47459 deg √[SUM (ERRORS)²]=4456.22739218
ITERATION  #59: A=6377.12734 km Ecc=.71000 ω=138.56418 deg √[SUM (ERRORS)²]=2377.60300719
ITERATION  #60: A=6377.30187 km Ecc=.81000 ω=213.83595 deg √[SUM (ERRORS)²]=3887.10052500
ITERATION  #61: A=6377.47640 km Ecc=.75529 ω=225.56000 deg √[SUM (ERRORS)²]=1250.86099995
ITERATION  #62: A=6277.47640 km Ecc=.71739 ω=224.33091 deg √[SUM (ERRORS)²]=0460.12395739
ITERATION  #63: A=6177.47640 km Ecc=.71899 ω=224.13317 deg √[SUM (ERRORS)²]=0330.20472467
ITERATION  #64: A=6077.47640 km Ecc=.72094 ω=224.66098 deg √[SUM (ERRORS)²]=0217.45002740
ITERATION  #65: A=6002.18338 km Ecc=.72299 ω=224.86027 deg √[SUM (ERRORS)²]=0080.15408107
ITERATION  #66: A=6022.73748 km Ecc=.72463 ω=225.13766 deg √[SUM (ERRORS)²]=0020.36231904
ITERATION  #67: A=6016.76933 km Ecc=.72419 ω=224.90929 deg √[SUM (ERRORS)²]=0008.57612011
ITERATION  #68: A=6020.10635 km Ecc=.72433 ω=225.05943 deg √[SUM (ERRORS)²]=0005.34985398
ITERATION  #69: A=6017.80717 km Ecc=.72425 ω=224.96052 deg √[SUM (ERRORS)²]=0003.48845386
ITERATION  #70: A=6019.36574 km Ecc=.72430 ω=225.02634 deg √[SUM (ERRORS)²]=0002.32481090
ITERATION  #71: A=6018.32287 km Ecc=.72427 ω=224.98242 deg √[SUM (ERRORS)²]=0001.55106925
ITERATION  #72: A=6019.01815 km Ecc=.72429 ω=225.01173 deg √[SUM (ERRORS)²]=0001.03537443
ITERATION  #73: A=6018.55427 km Ecc=.72427 ω=224.99218 deg √[SUM (ERRORS)²]=0000.69057852
ITERATION  #74: A=6018.86368 km Ecc=.72429 ω=225.00522 deg √[SUM (ERRORS)²]=0000.46077383
ITERATION  #75: A=6018.65724 km Ecc=.72428 ω=224.99652 deg √[SUM (ERRORS)²]=0000.30736057
ITERATION  #76: A=6018.79495 km Ecc=.72428 ω=225.00232 deg √[SUM (ERRORS)²]=0000.20506258
ITERATION  #77: A=6018.70308 km Ecc=.72428 ω=224.99845 deg √[SUM (ERRORS)²]=0000.13679601
ITERATION  #78: A=6018.76437 km Ecc=.72428 ω=225.00103 deg √[SUM (ERRORS)²]=0000.09126299
ITERATION  #79: A=6018.72348 km Ecc=.72428 ω=224.99931 deg √[SUM (ERRORS)²]=0000.06088258
ITERATION  #80: A=6018.75075 km Ecc=.72428 ω=225.00046 deg √[SUM (ERRORS)²]=0000.04061689
ITERATION  #81: A=6018.73256 km Ecc=.72428 ω=224.99969 deg √[SUM (ERRORS)²]=0000.02709631
ITERATION  #82: A=6018.74470 km Ecc=.72428 ω=225.00021 deg √[SUM (ERRORS)²]=0000.01807675
ITERATION  #83: A=6018.73660 km Ecc=.72428 ω=224.99986 deg √[SUM (ERRORS)²]=0000.01205941
ITERATION  #84: A=6018.74200 km Ecc=.72428 ω=225.00009 deg √[SUM (ERRORS)²]=0000.00804516
ITERATION  #85: A=6018.73840 km Ecc=.72428 ω=224.99994 deg √[SUM (ERRORS)²]=0000.00536712
ITERATION  #86: A=6018.74080 km Ecc=.72428 ω=225.00004 deg √[SUM (ERRORS)²]=0000.00358055
ITERATION  #87: A=6018.73920 km Ecc=.72428 ω=224.99997 deg √[SUM (ERRORS)²]=0000.00238867
ITERATION  #88: A=6018.74027 km Ecc=.72428 ω=225.00002 deg √[SUM (ERRORS)²]=0000.00159355
ITERATION  #89: A=6018.73955 km Ecc=.72428 ω=224.99999 deg √[SUM (ERRORS)²]=0000.00106309
ITERATION  #90: A=6018.74003 km Ecc=.72428 ω=225.00001 deg √[SUM (ERRORS)²]=0000.00070922
ITERATION  #91: A=6018.73971 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00047314
ITERATION  #92: A=6018.73992 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00031564
ITERATION  #93: A=6018.73978 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00021057
ITERATION  #94: A=6018.73988 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00014048
ITERATION  #95: A=6018.73981 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00009372
ITERATION  #96: A=6018.73986 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00006252
ITERATION  #97: A=6018.73983 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00004171
ITERATION  #98: A=6018.73985 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00002783
ITERATION  #99: A=6018.73983 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00001856
ITERATION #100: A=6018.73984 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00001238
ITERATION #101: A=6018.73984 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00000826
ITERATION #102: A=6018.73984 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00000551
ITERATION #103: A=6018.73984 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00000368
ITERATION #104: A=6018.73984 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00000245
ITERATION #105: A=6018.73984 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00000164
ITERATION #106: A=6018.73984 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00000109
ITERATION #107: A=6018.73984 km Ecc=.72428 ω=225.00000 deg √[SUM (ERRORS)²]=0000.00000073

CONVERGED SOLUTION: A=6018.73984 km Ecc=.72428 Io=.00000 deg ω=225.00000 deg Ω=.00000 deg
                    √[SUM (ERRORS)²]=.00000073
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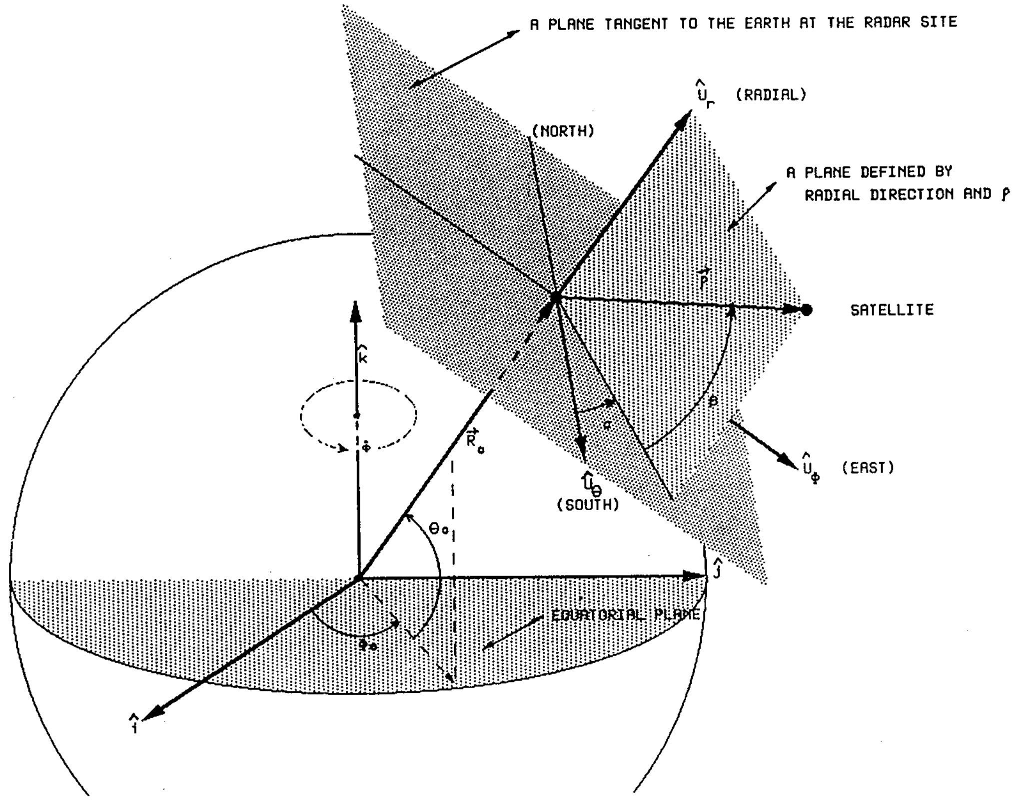

Figure 1. Coordinate systems showing azimuth (α) and elevation (β).

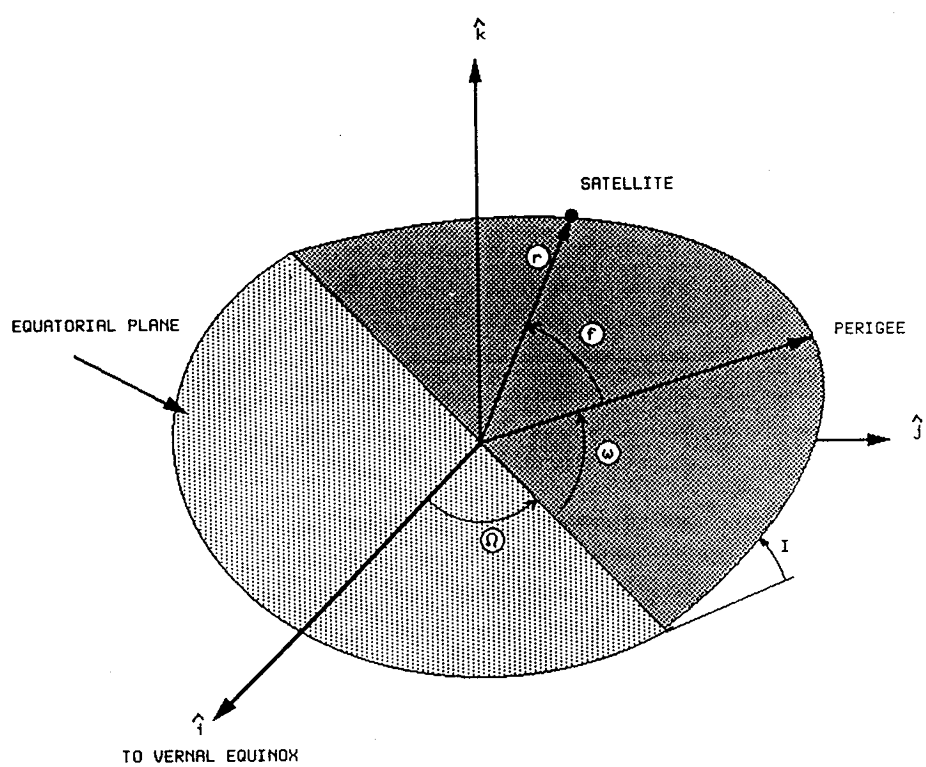

Figure 2. Orbital elements.

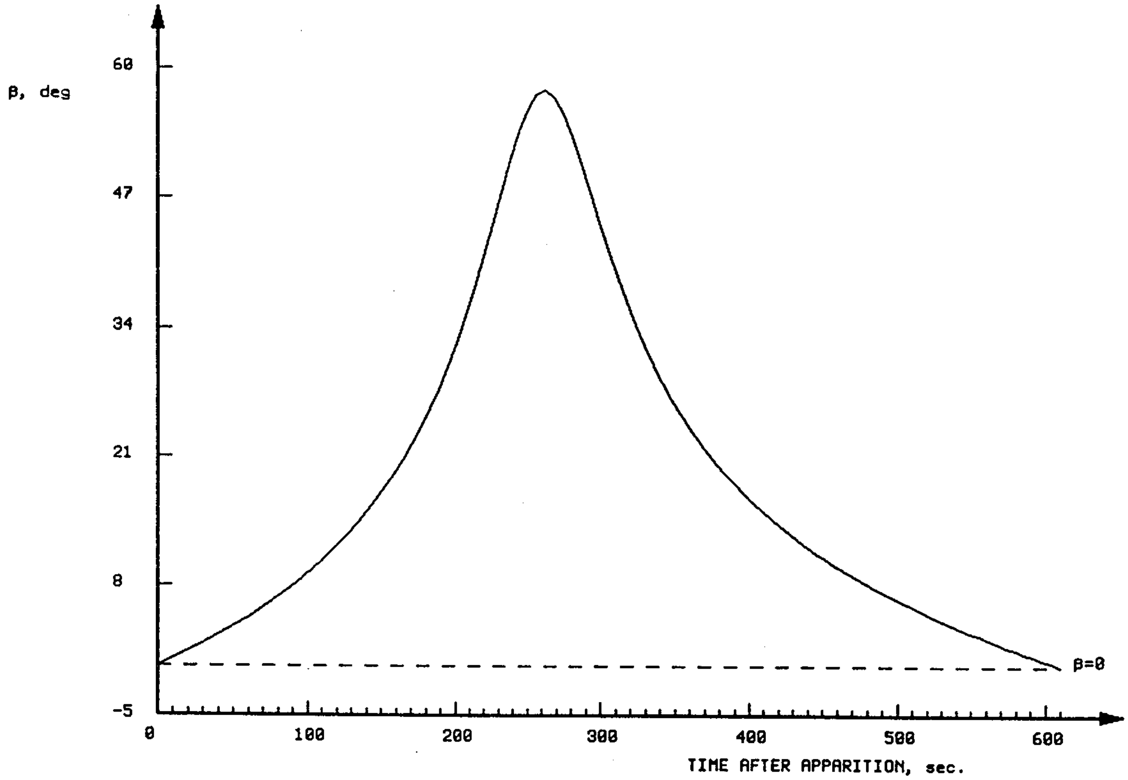

Figure 3. β versus time for the example from table 1 (satellite).

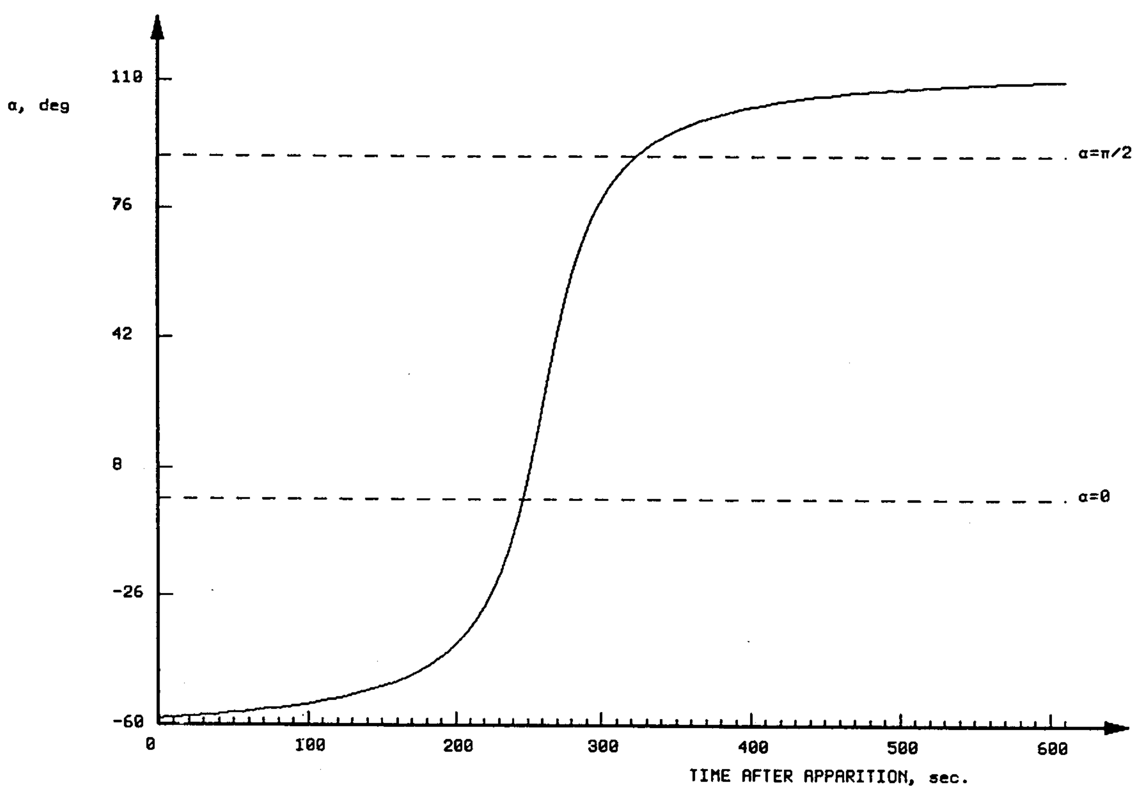

Figure 4. α versus time for the example from table 1 (satellite).

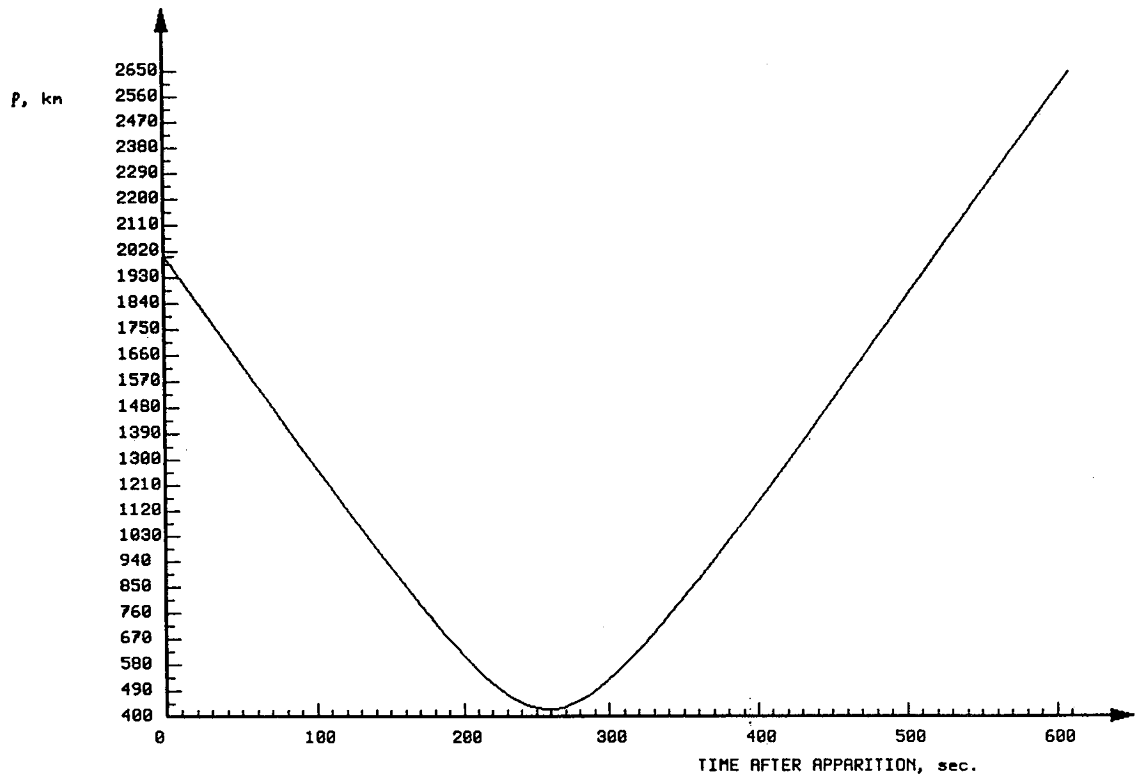

Figure 5. ρ versus time for the example from table 1 (satellite).

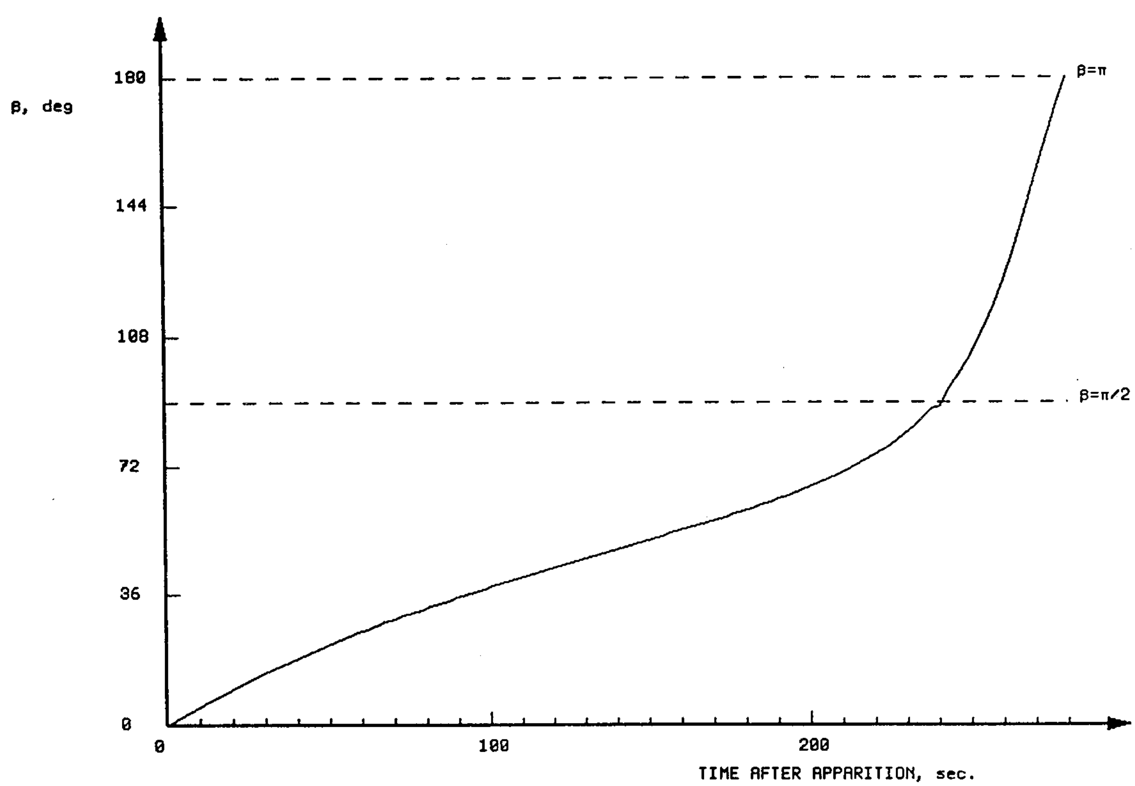

Figure 6. β versus time for the example from table 4 (ICBM).

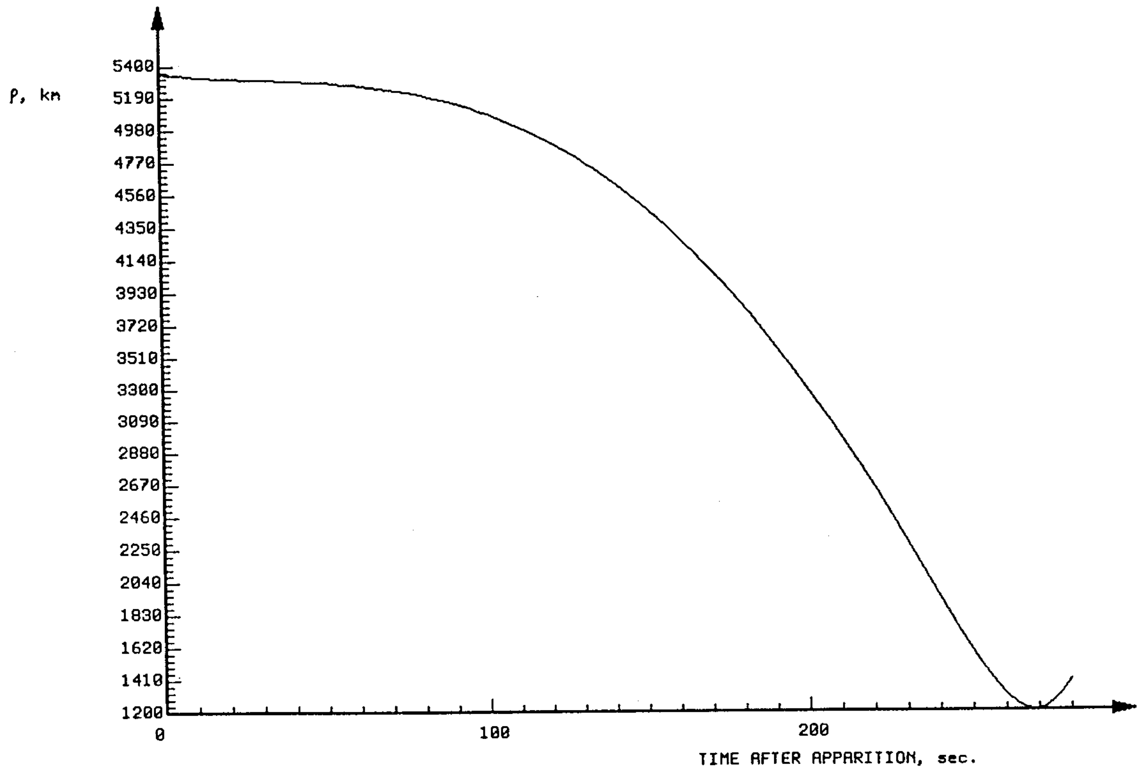

Figure 7. ρ versus time for the example from table 4 (ICBM).

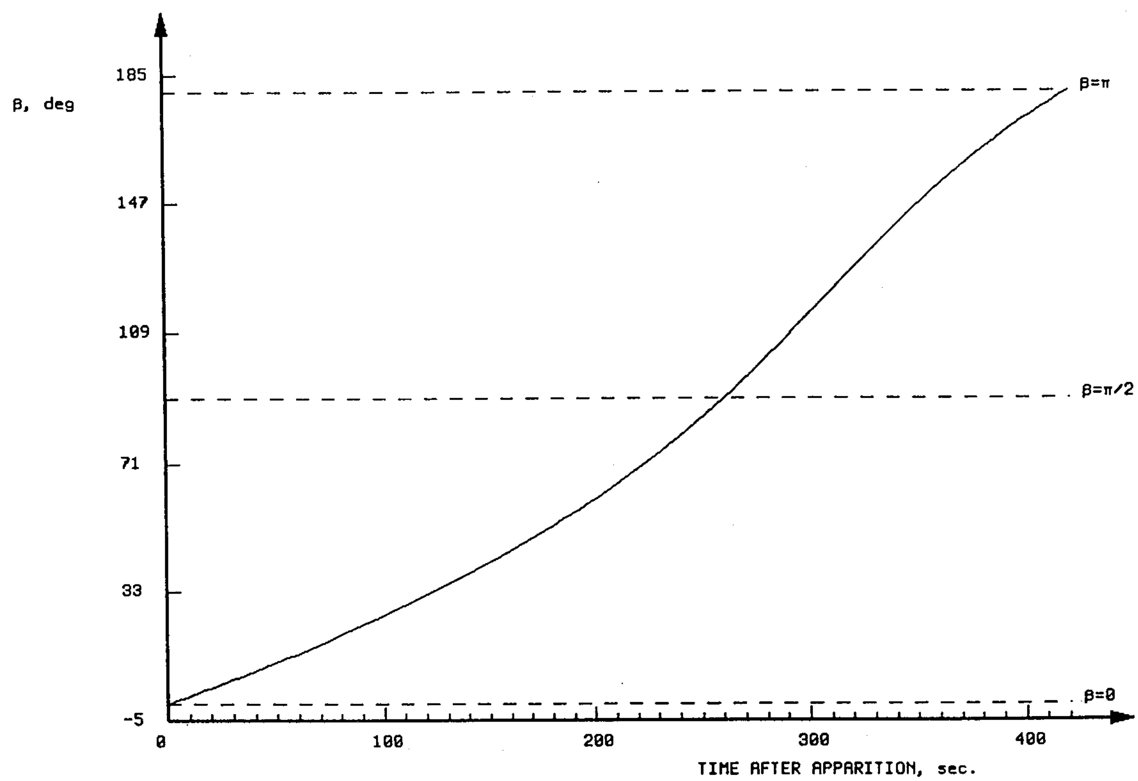

Figure 8. β versus time for 200 km range missile.

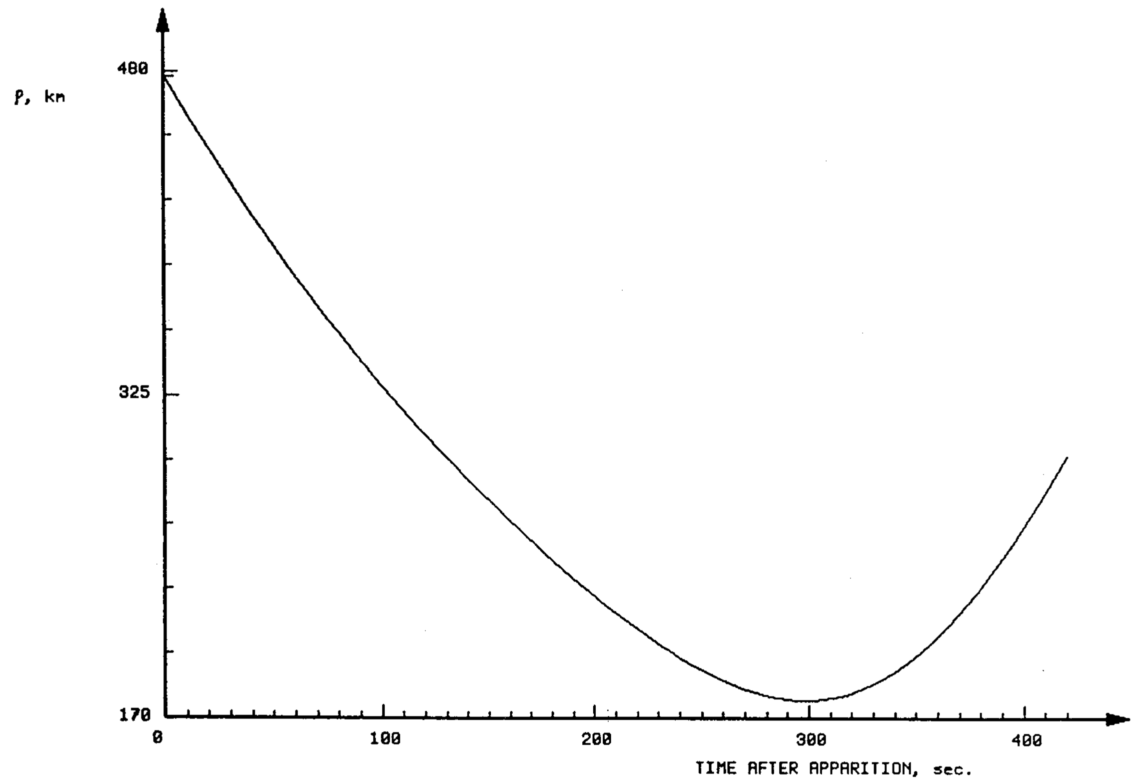

Figure 9. ρ versus time for 200 km range missile.

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1. Escobal, P.R.: "Method of Orbit Determination," John Wiley and Sons, New York, 1965.

2. McCuskey, S.W.: "Introduction to Celestial Mechanics," Addison-Wesley Publishing Co., Reading, Mass., 1963.

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A single observation station, located at an arbitrary point on the surface of the Earth, can determine only the azimuth and elevation angles of a satellite or ballistic vehicle, and the time at which these observations occur. No information is available about the range or the range-rate of the target. It is shown that five observations of either the elevation or the azimuth, and the time of either set of observations, determine the complete set of orbital elements of the target. The implementation of the theory presented here could provide a great reduction in the hardware costs associated with satellite and reentry vehicle tracking.

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